Electric Networks (ELE 302): Notes

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Introduction

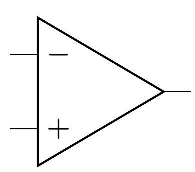
This is my compilation of notes from Electric Networks (ELE 302) from Ryerson University. All information comes from my professor's lectures, the textbook *Fundamentals of Electric Circuits*, and online resources.

Chapter 5: Operational Amplifiers

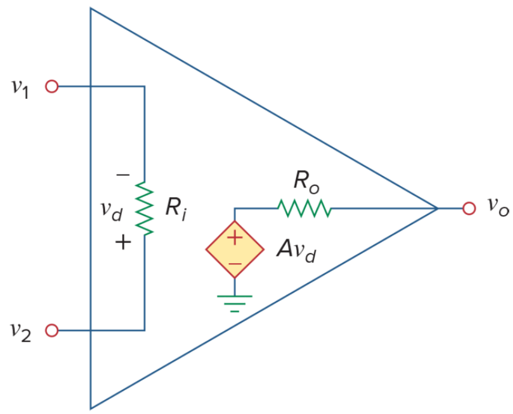
The *operational amplifier* or *op amp* for short is an active circuit element (meaning it supplies energy to the circuit) which acts as a voltage controlled voltage source.

An op amp takes in two signals (through two pins) and does some *operation* to them, then *amplifies* it, and then returns the signal through a new pin. Operations that an op amp can do include: addition, subtractions, multiplication, division, differentiation, and integration. Op amps are made of a complex arrangement of transistors, we will consider the op amp to be a circuit element and not concern ourselves with what exactly inside.

The symbol for an op amp is:



You can think of the op amp as this equivalent circuit:

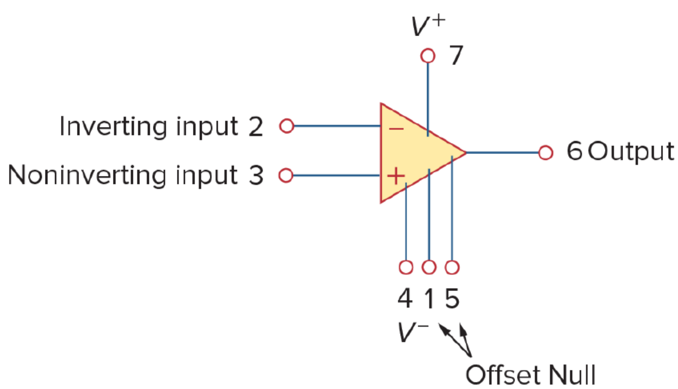


As you can see, the output voltage is controlled by the voltage controlled voltage source. The voltage is v_d which is the difference between v_2 and v_1, multiplied by some A which is the amplification multiplier (also called *open loop gain*).

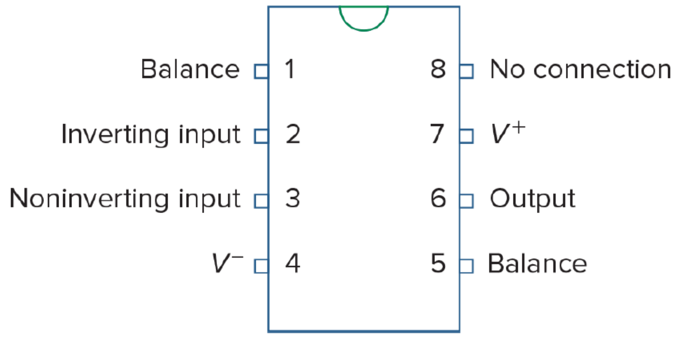
$$v_o \approx Av_d = A(v_2 - v_1)$$

It is approximate since there is a resistor there with a value R_o, however ideally that resistance is negligible. More on this later.

An op amp is an active element which means it needs to be powered, this is usually omitted in circuit diagrams but cannot be forgotten in practical applications. A diagram including the power would be as follows:



Where the numbers refer to this diagram of a real op amp:



We do not care about pins 1,5, or 8. Pins 4 and 7 handle active power. Pins 2 and 3 hand input, and pin 6 handles output. If you apply a voltage to the inverting input, it will appear with the opposite sign in the output, if you apply a voltage to the non-inverting input it will appear with the same sign in the output.

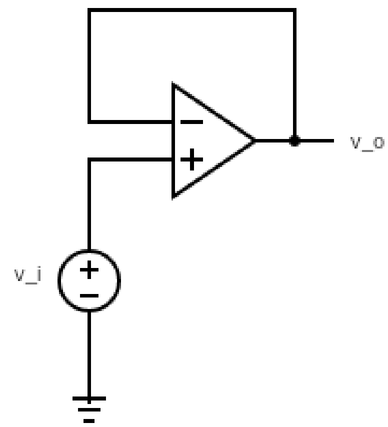
Next we compare ideal versus real values seen for the elements within the op amp:

Variable	**Ideal**	**Real**
A	∞	$10^5 \to 10^8$
R_i	$\infty\Omega$	$10^5 \to 10^{13}\Omega$
R_o	0Ω	$10 - 100\Omega$

The ideal R_i is infinite, this is because as $R_i \to \infty \implies i_{in} \to 0$. We want the input current to be zero to not change the input signals. A is called the *open loop feedback* due to the configuration. We will now discuss *closed loop feedback* configurations.

Negative Feedback Configuration of an Op Amp

You can connect the output of the op amp to the negative input, and get a circuit such as the following:



One interesting fact we usually want to know about configurations of op amps is the ratio $\frac{v_o}{v_i}$. We can solve for this:

$$v_o = A(v_i - v_o)$$

$$\frac{v_o}{v_i} = \frac{A}{A+1}$$

But remember $A \to \infty$, so:

$$\frac{v_o}{v_i} = \lim_{A\to\infty} \frac{A}{A+1}$$

$$\frac{v_o}{v_i} = 1 \implies v_o = v_i$$

Generally you can think of this configuration as a negative feedback loop, which brings v_o closer and closer to v_i every unit of time, using the equation:

$$v_{o(new)} = A(v_i - v_{o(old)})$$

Assumptions

For an idea op amp in this negative feedback configuration we can make the following assumptions:

1. The input current to either pin is zero.

$$i_1 = i_2 = 0$$

2. The voltage difference across the input pins is zero.

$$v_2 - v_1 = 0 \implies v_2 = v_1$$

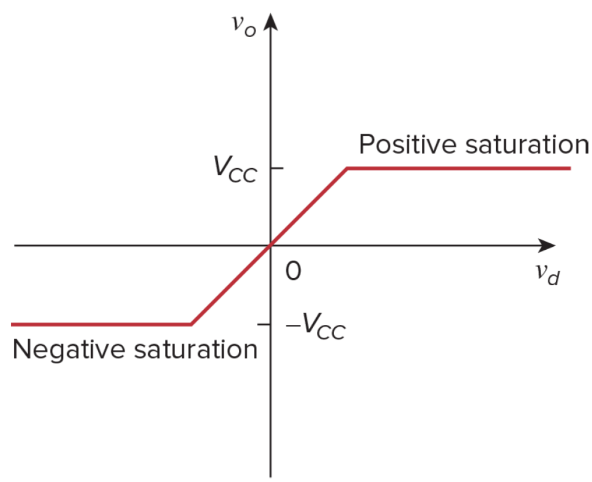
These two equations will let you solve problems using an op amp in this configuration.

There is a practical limitation of real op amps which is important for lab 1. Remember that this component needs to be powered by an outside source, and so the output voltage cannot exceed the input voltage that powers the entire circuit block. This voltage is called V_{CC}, and so:

$$-V_{CC} \leq v_o \leq V_{CC}$$

This means that the output of the op amp (v_o) can run in three modes in relation to V_{CC}:

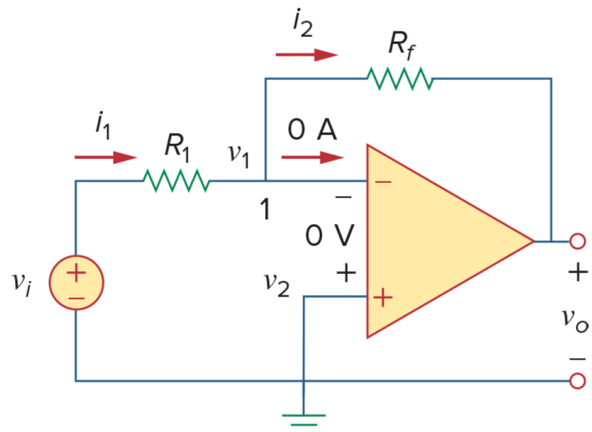
1. Positive saturation $v_o = V_{CC}$
2. Negative saturation $v_o = -V_{CC}$
3. Linear region $-V_{CC} \leq v_o \leq V_{CC}$



Specific Configurations of the Op Amp

Inverting Amplifier

In this configuration, the noninverting input is grounded, while the other input is connected to the voltage source though a resistor. There is a *negative feedback loop* with a resistor. The goal is to know how v_o relates to v_i.



Through the use of:

$$v_1 = v_2$$

$$i_a = i_b = 0$$

Along with *KCL* at the inverting input:

$$i_1 = i_2$$

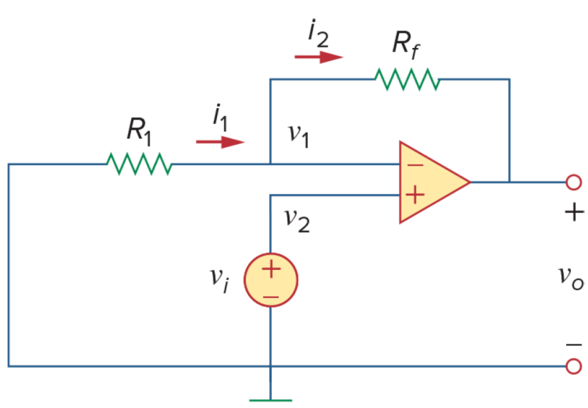
We can derive that:

$$\boxed{v_o = -\frac{R_f}{R_1} v_i}$$

You can control the value of v_o by adjusting R_f and R_1. Notably, the output voltage will be of the opposite sign of the input, as in it was *inverted*.

Non-Inverting Amplifier

In this configuration v_i is applied the noninverting input, with a resistor between the ground and the inverting input. There is a *negative feedback loop* with a resistor. The goal is to know how v_o relates to v_i.



Through the use of:

$$v_1 = v_2$$

$$i_a = i_b = 0$$

Along with *KCL* at the inverting input:

$$i_1 = i_2$$

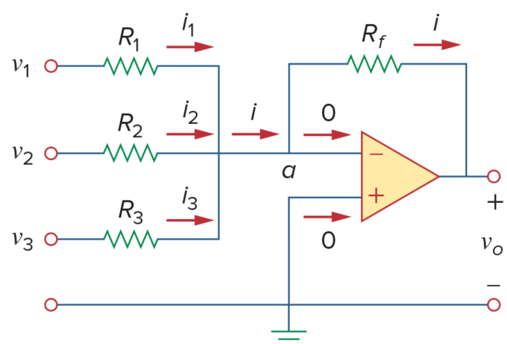
We can derive that:

$$\boxed{v_o = (1 + \frac{R_f}{r_1})v_i}$$

You can control the value of v_o by adjusting R_f and R_1.

Summing Amplifier

In this configuration, three voltages are applied v_1, v_2, v_3, and the output is the sum of the three inputs. There is *negative feedback loop* with a resistor.



Through the use of:

$$v_1 = v_2$$

$$i_a = i_b = 0$$

Along with *KCL* at the inverting input:

$$i_1 + i_2 + i_3 - i = 0$$

We can derive that:

$$\boxed{v_o = -\left(\frac{R_f}{R_1}v_1 + \frac{R_f}{R_2}v_2 + \frac{R_f}{R_3}v_3\right)}$$

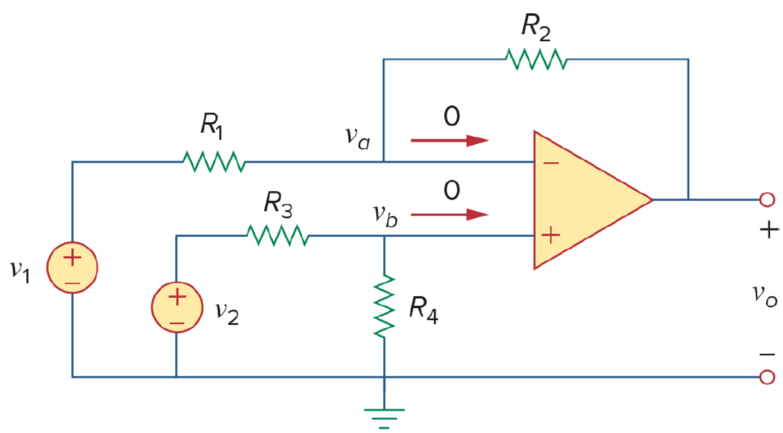
This is called the *weighted sum* of all the voltages, as it takes into account each voltage's resistance. In the case that$R_1 = R_2 = R_3 = R_f$ then:

$$v_o = -(v_1 + v_2 + v_3)$$

This extends to the sum of n voltages.

Difference Amplifier

In this configuration, the difference between two inputs are amplified. There is a *negative feedback loop* with a resistor.



Through the use of:

$$v_1 = v_2$$

$$i_a = i_b = 0$$

Along with *KCL* at note b, we can derive that:

$$\boxed{v_o = \frac{R_2}{R_1}(v_b - v_a)}$$

... and if $R_2 = R_1$:

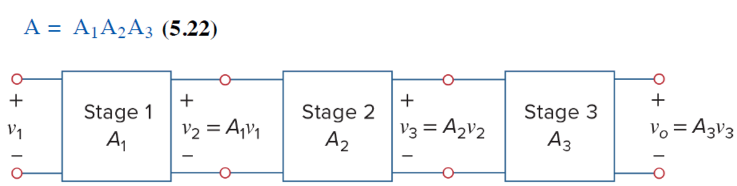
$$v_o = v_b - v_a$$

... and it would be called a *subtractor*.

This is different than a regular op amp with no negative feedback because now you can control the amplification by controlling $\frac{R_2}{R_1}$.

Cascading Amplifiers

You can put op amps that feed into each other in a line in whatever configuration you want. Let's say you have three op amps in a row, which feed into each other, all with some amplification factor of A, like the following image:



Then we know that:

$$v_0 = A_1 A_2 A_3 v_1$$

Which is true generally for n cascaded op amps.

Chapter 8: Second Order Circuits

In this chapter we study second order circuits, which are circuits characterized by a second-order differential equation. Similar to how we did first order circuits in ELE202, here second order circuits are interested in those with two energy-storage elements (inductors and capacitors).

Remember that in complex circuits, you only need to go through the whole process of finding one of the voltages or currents in the circuit, and from there you can deduce the rest using more elementary methods.

8.2 Finding Initial and Final Values

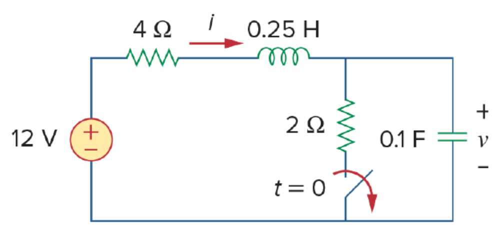
DEs in this course will always be initial value problems, as we want to describe how a circuit is changing over time, given some initial event. We also need to know end behavior.

In second order circuits, you need to know the initial values of $i(t), v(t)$ but also $i'(t), v'(t)$, depending on which circuit variable you are trying to track.

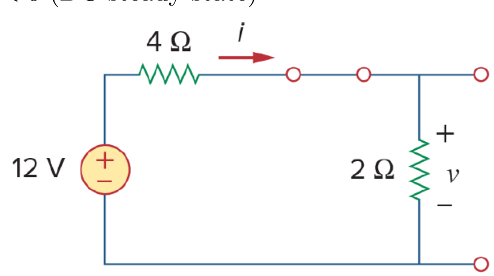
Unless otherwise mentioned, v denotes capacitor voltage, and i denotes inductor current.

The best way to find these values in a second order circuit is to draw the circuit in all of it's states:

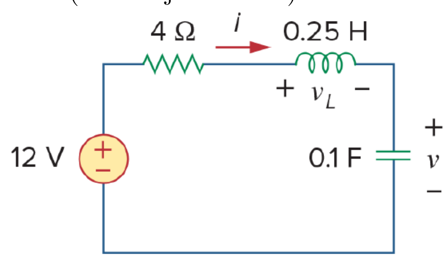
1. Given diagram.



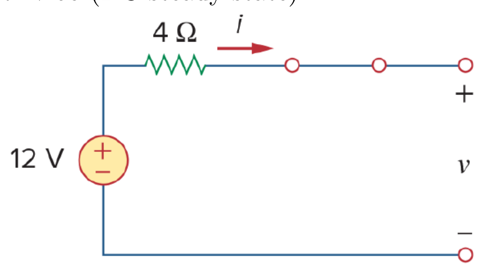
2. $t < 0$ (DC steady state)



3. $t = 0$ (Switch just closed)



4. $t \to \infty$ (DC steady state)



Recall that over a capacitor:

$$\boxed{v(0^+) = v(0^-)}$$

As in the voltage is continuous. Or in an inductor:

$$\boxed{i(0^+) = i(0^-)}$$

As in the current is continuous. This means you can find the initial values of either current of voltage by finding those values before the event at time $t = 0$ (before the switch closes). This is the state 2 diagram above.

Recall that for a capacitor, the following is true:

$$\boxed{i_C(t) = C\frac{dv}{dt}}$$

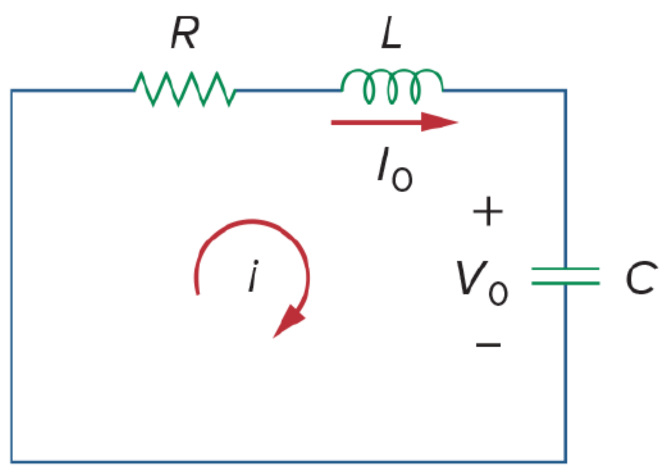
... and for an inductor:

$$\boxed{v_L(t) = L\frac{di}{dt}}$$

You can use these equations to solve for $v'(0)$ and $i'(0)$. Make sure that you find the correct current and voltage values.

8.3 The Source-Free Series RLC Circuit

In this section we analyze the following circuit:



Notably this circuit has no independent sources, and so all of its energy is that which is stored inside the inductor and capacitor at time $t = 0$, meaning:

$$v(0) = V_0$$

$$i(0) = I_0$$

Applying KVL around the loop we get:

$$v_{resistor} + v_L + v_C = 0$$

We will choose to write everything in terms of the current, recalling that:

$$v_{resistor} = Ri$$

$$v_L = L\frac{di}{dt}$$

$$i_C = C\frac{dv}{dt} \implies v(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau$$

... which gives us:

$$Ri + L\frac{di}{dt} + \frac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau$$

... differentiating both sides with respect to time to get rid of the integral:

$$Ri' + Li'' + \frac{i}{C} = 0$$

... and putting it in standard form:

$$\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{LC}i = 0$$

This is a second order LDE which can have three possible types of solutions.

The solution to this DE depends on the solutions to it's characteristic polynomial (s_1, s_2):

$$s_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

$$s_1 = -\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

These solutions are measured in nepers per second (Np/s).

We define (for source-free series RLC circuits):

$$\alpha = \frac{R}{2L}$$

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

... where α is called the *neper frequency* and ω_0 is called the *resonant frequency*.

The particular values of α and ω_0 change the way the variables will change over time, there are three cases:

1. *Overdamped* when $\alpha > \omega_0$
2. *Critically damped* when $\alpha = \omega_0$
3. *Underdamped* when $\alpha < \omega_0$

You can find $i'(0)$ by using the following equation:

$$Ri(0) + L\frac{di(0)}{dt} + V_0 = 0 \implies \frac{di(0)}{dt} = -\frac{1}{L}(RI_0 + V_0)$$

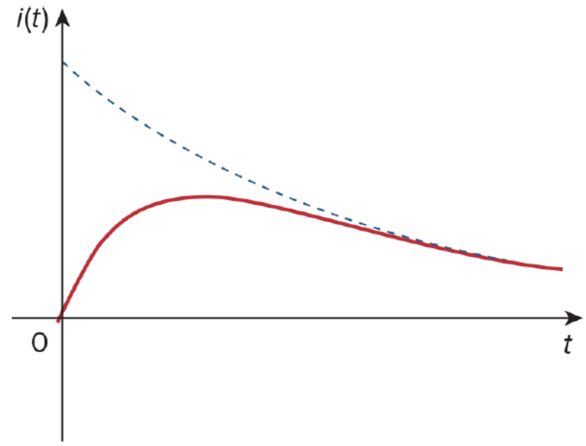
Overdamped

In this case our solution to the DE is:

$$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

Where you can find the values of A_1, A_2 given $i(0)$ and $i'(0)$.

Both of our roots are negative and real, meaning the function will decay as $t \to \infty$:

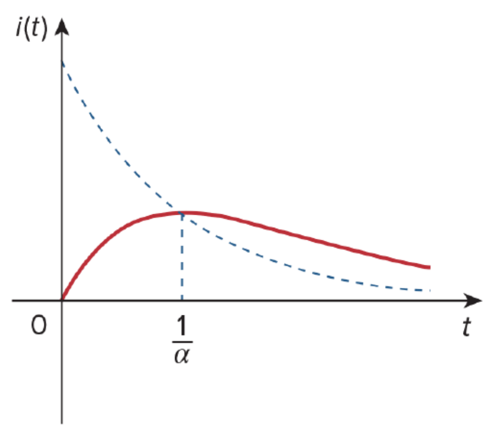


Critically Damped

In this case our solution to the DE is:

$$i(t) = (A_1 + A_2 t)e^{-\alpha t}$$

Where you can find the values of A_1, A_2 given $i(0)$ and $i'(0)$.

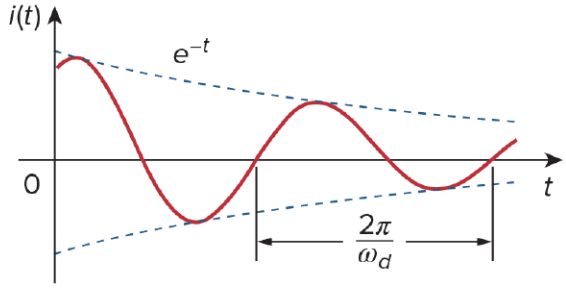


Underdamped

In this case our solution to the DE is:

$$i(t) = e^{-\alpha t}(A_1 \cos(\omega_d t) + A_2 \sin(\omega_d t))$$

... where $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$ and you can find A_1, A_2 given $i(0)$ and $i'(0)$.

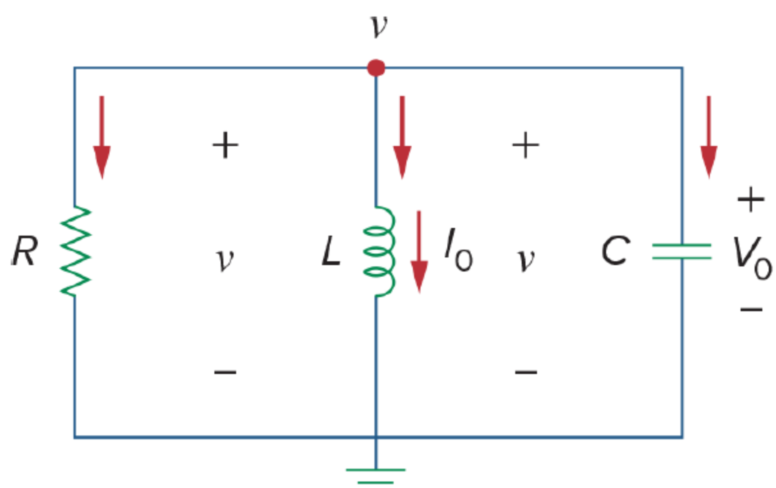


The behavior of this network is because of the damping effect, which is the gradual loss of the initial stored energy. This happens because of the resistor, in fact controlling the resistor can change what state the network is in. If $R = 0$ and all elements ideal then the response would be a perfect sinusoid, and called *loss-less*. This oscillatory response happens because of the two different types of storage elements in the circuit which flow the energy back and forth between the inductor and capacitor.

The critically damped case is the borderline between the underdamped and the overdamped cases, and it decays the fastest. In most practical applications we seek an overdamped circuit that is as close as possible to the critically damped case.

8.4 The Source-Free Parallel RLC Circuit

In this section we analyze the following circuit:



Notably this circuit has no independent sources, and so all of its energy is that which is stored inside the inductor and capacitor at time $t = 0$, meaning:

$$v(0) = V_0$$

$$i(0) = I_0$$

Applying KCL at the top node we get the following differential equation (in terms of $v(t)$):

$$\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{1}{LC}v = 0$$

The characteristic polynomial of this DE has solutions of:

$$s_{1,2} = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

We define (for source-free parallel RLC circuits):

$$\alpha = \frac{1}{2RC}$$

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

Once again, the particular values of α and ω_0 change the way the variables will change over time, there are three cases:

1. *Overdamped* when $\alpha > \omega_0$
2. *Critically damped* when $\alpha = \omega_0$
3. *Underdamped* when $\alpha < \omega_0$

In all cases, constants A_1, A_2 can be determined from the initial conditions of $v(0)$ and $v'(0)$, which we can get from the following KVL equation:

$$\frac{V_0}{R} + I_0 + C\frac{dv(0)}{dt} \implies \frac{dv(0)}{dt} = -\frac{V_0 + RI_0}{RC}$$

Overdamped

In this case the solution is:

$$v(t) = A_1e^{s_1t} + A_2e^{s_2t}$$

Critically Damped

In this case the solution is:

$$v(t) = (A_1 + A_2t)e^{-\alpha t}$$

Underdamped

In this case the solution is:

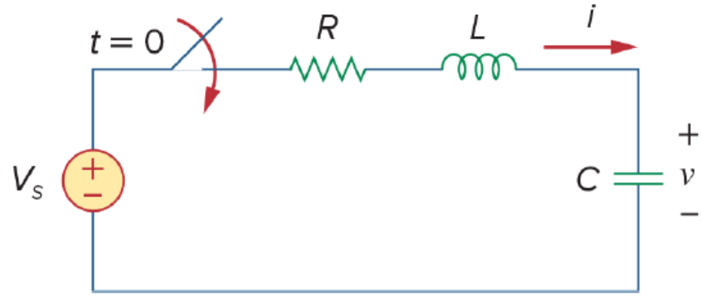
$$v(t) = e^{-alphat}(A_1\cos(\omega_d t) + A_2\sin(\omega_d t))$$

Where:

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2}$$

8.5 Step Response of a Series RLC Circuit

The step response of a circuit is obtained by a sudden application of a DC source by means of a switch. In this section we study the following circuit:



Applying KVL around the loop when $t > 0$, we get the following equation:

$$\frac{d^2v}{dt^2} + \frac{R}{L}\frac{dv}{dt} + \frac{v}{LC} = \frac{V_s}{LC}$$

Which is a non-homogenous LDE, whose solution can be written in the form:

$$v(t) = v_t(t) + v_{ss}(t)$$

$v_t(t)$ is called the *transient response* and is the same as the solution to the *source-free RLC series circuit*. v_{ss} is called the *steady state response* and is the final value of $v(t)$, as in:

$$v_{ss}(t) = v(\infty)$$

Which in our case:

$$v_{ss}(t) = V_s$$

For this reason our solutions to the second order RLC circuit are:

- Overdamped:
$$v(t) = V_s + A_1e^{s_1t} + A_2e^{s_2t}$$
- Critically Damped:
$$v(t) = V_s + (A_1 + A_2t)e^{-\alpha t}$$
- Underdamped:
$$v(t) = V_s + (A_1\cos(\omega_dt) + A_2\sin(\omega_dt))e^{-\alpha t}$$

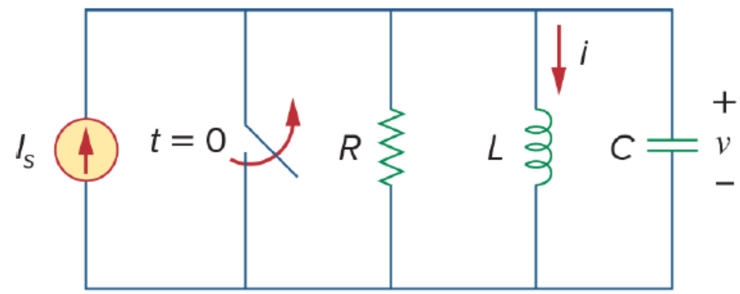
More generally, the complete response for any variable $x(t)$ can be found directly through:

$$x(t) = x_t(t) + x_{ss}(t)$$

Where $x_t(t)$ is the transient response which dies out over time, and $x_{ss}(t) = x(\infty)$ and represents the steady state response of the variable.

8.6 Step Response of a Parallel RLC Circuit

In this section we study the following circuit:



Applying KCL at the top node for $t > 0$ gets us the differential equation:

$$\frac{d^2i}{dt^2} + \frac{1}{RC}\frac{di}{dt} + \frac{1}{LC} = \frac{I_s}{LC}$$

Which is a non-homogenous LDE¡ whose solution can be written in the form:

$$i(t) = i_t(t) + i_{ss}(t)$$

$i_t(t)$ is called the *transient response* and is the same as the solution to the *source-free RLC parallel circuit*. i_{ss} is called the *steady state response* and is the final value of $i(t)$, as in:

$$i_{ss}(t) = i(\infty)$$

Which in our case:

$$i_{ss}(t) = I_s$$

For this reason our solutions to the second order RLC circuit are:

- Overdamped:
$$i(t) = I_s + A_1 e^{s_1 t} + A_2 e^{s_2 t}$$
- Critically damped:
$$i(t) = I_s + (A_1 + A_2 t)e^{-\alpha t}$$
- Underdamped:
$$i(t) = I_s + (A_1 \cos(\omega_d t) + A_2 \sin(\omega_d t))e^{-\alpha t}$$

8.7 General Second-Order Circuits

This section describes a method to solve a general RLC circuit for a general variable x:

1. Determine the initial conditions:
$$x(0), x'(0)$$

2. Turn off all independent sources and find the form of the transient response $x_t(t)$ by applying KCL and KVL. Solve the DE, with two unknown constants, in one of the following forms: overdamped, critically damped, underdamped.

3. Obtain the steady-state response:

$$x_{ss}(t) = x(\infty)$$

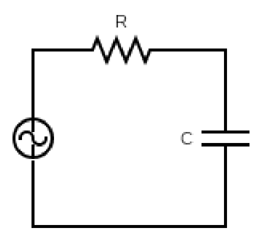
4. The total response is now found as the sum of the transient response and steady-state response:

$$x(t) = x_t(t) + x_{ss}(t)$$

5. Determine constants associated with the transient response by applying the initial conditions.

Chapter 14: Frequency Response

When doing analysis of AC circuits, we have kept the *frequency* of the sinusoid constant at some value ω. In this chapter we analyse the response some element in the circuit has as a function of the change in the input variable's *frequency parameter*. We call this the circuit's *frequency response*. Take for example the following circuit:



If the source voltage is in the form:

$$V(t) = A\cos(\omega t + \phi)$$

... then the voltage across the capacitor would also be in the form:

$$V_C(t) = A_0 \cos(\omega t + \phi_0)$$

If we vary the frequency, the amplitude and phase shift of V_C would change as well. This is because:

$$\bar{V_C}(\omega) = \frac{1}{\hat{j}\omega C}$$

... which is a function of ω.

14.2 The Transfer Function

The transfer function is a complex valued function which is the analytical tool used to study the frequency response of a circuit.

The transfer function is the frequency-dependent ratio between a forced function and its forc*ing* function. For example a the voltage across a resistor would be a *forced function*, while the voltage source would be its *forcing function.*

The *forcing function* (also called the input) is denoted $\bar{X}(\omega)$. The *forced function* (also called the output) is denoted $\bar{Y}(\omega)$. By definition the transfer function ($\bar{H}(\omega)$) is:

$$\boxed{\bar{H}(\omega) = \frac{\bar{Y}(\omega)}{\bar{X}(\omega)}}$$

Sometimes this function is denoted $\bar{H}(\hat{\jmath}\omega)$ since the ω and $\hat{\jmath}$ will always be together, and this notation emphasizes that it is a complex defined function.

There are *four* possible transfer functions:

1. **Voltage Gain**:
$$\bar{H}(\omega) = \frac{\bar{V}_0(\omega)}{\bar{V}_i(\omega)}$$
2. **Current Gain**:
$$\bar{H}(\omega) = \frac{\bar{I}_0(\omega)}{\bar{I}_i(\omega)}$$
3. **Transfer Impedance**:
$$\bar{H}(\omega) = \frac{\bar{V}_0(\omega)}{\bar{I}_i(\omega)}$$
4. **Transfer Admittance**:
$$\bar{H}(\omega) = \frac{\bar{I}_0(\omega)}{\bar{V}_i(\omega)}$$

Note that the output of the transfer function is a complex value and so it can be graphed as a vector on the complex plane. As ω varies, the vector will both rotate ($\phi(\omega)$) and scale $H(\omega)$, and written in the form:

$$\bar{H}(\omega) = H(\omega)\angle\phi(\omega)$$

This means that the plot of $\bar{H}$ can be broken into two, one for magnitude and one for phase. We will learn how to plot approximations of these using the **Bode Plot** technique later on.

The transfer function will always be a ration of two complex polynomials $\bar{N}$ and $\bar{D}$:

$$\bar{H} = \frac{\bar{N}}{\bar{D}}$$

The roots of the *numerator polynomial* are called *zeros* ($\omega = z_1, z_2, \dots$) of the transfer function, and represent frequencies in which the forced function stops.

The roots of the *denominator polynomial* are called *poles* ($\omega = p_1, p_2, \dots$) of the transfer function, and represent frequencies in which the transfer function goes to infinity.

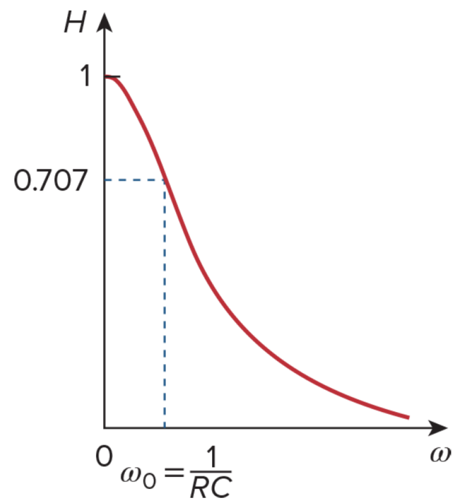
We now have the skills to determine the transfer function of the intro circuit (by current division):

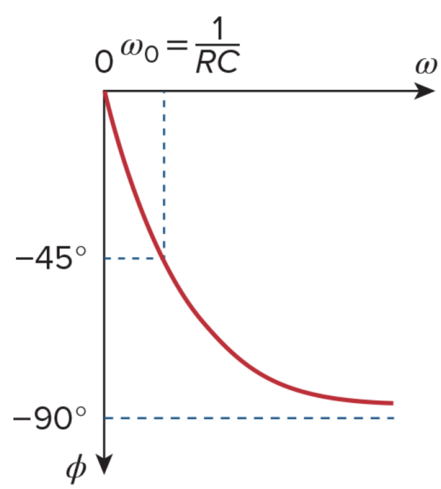
$$\bar{H}(\omega) = \frac{\bar{V}_0}{\bar{V}_s} = \frac{1\hat{\hat{j}}\omega C}{R + 1/\hat{\hat{j}}\omega C} = \frac{1}{1 + \hat{\hat{j}}\omega RC}$$

Which can be broken up into two real valued functions:

$$H(\omega) = \frac{1}{\sqrt{1 + (\omega RC)^2}}, \phi(\omega) = -\arctan(\omega RC)$$

Whose plots can be seen below:





We will learn how to plot these graphs in a systematic way by hand, but first we need to learn how gain is notated.

The Decibel Scale

The unit of measurement of *gain* between to variables is the *bel* defined as:

$$G_B = \log_{10} \frac{P_2}{P_1}$$

The *decibel* is the most commonly variant, which is just a tenth of a *bel*:

$$G_{dB} = 10 \log_{10} \frac{P_2}{P_1}$$

Recall that gain is the measure of how much bigger or smaller one variable is in comparison to another. One of the most useful properties of using the decibel scale is that if kdB represents an increasing gain, $-kdB$ represents a equivalent decreasing gain.

If the previous definition of gain is for power, the following two are for voltage and resistance:

$$G_{dB} = 20 \log_{10} \frac{V_2}{V_1}$$

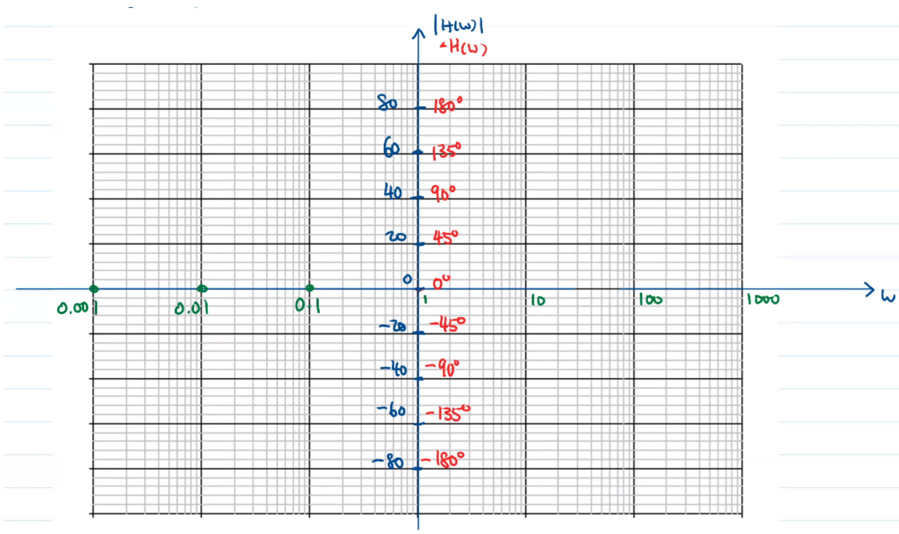
$$G_{dB} = 20 \log_{10} \frac{I_2}{I_1}$$

Decibels are used to describe the transfer function in voltage and current gain form as they are dimensionless.

Bode Plots

In this section we describe a systematic method to sketch an approximation to to magnitude versus frequency, and phase versus frequency plots of the transfer function. This is called the *method of Bode plots*, or just *Bode plots*.

The *magnitude* in decibels is plotted against the logarithm of the frequency, and on a separate plot the phase in degrees is plotted against the logarithm of the frequency. The setup would look like the following:



Here, the *red* represents the plot for the phase, while the *blue* markings represent the plot for the magnitude. This will typically be done on two plots, but it is combined here to save space.

The transfer function will always be written in the form:

$$H(w) = \frac{K(\jmath\omega)^{\pm 1}(1+\frac{\jmath\omega}{z_1})\left(1+\frac{\jmath 2\beta_1\omega}{\omega_k}+\left(\frac{\jmath\omega}{\omega_k}\right)^2\right)\cdots}{(1+\frac{\jmath\omega}{p_1})\left(1+\frac{\jmath 2\omega}{\omega_n}+\left(\frac{\jmath\omega}{\omega_n}\right)^2\right)\cdots}$$

This is called *standard form* of the transfer function and will be the basis of how we plot Bode plots. The idea is that we learn how to plot each factor separately, and then add them all together to make the entire plot.

Terminology:

1. Roots of the numerator are called **zeros**.
2. Roots of the denominator are called **poles.**

The dots refer to higher and higher degree terms which can appear in the transfer function, for this course we will only have to consider constant, linear, and quadratic terms.

The reason we use semilog paper is that it is a lot easier to graph the log (base 10 unless otherwise specified) of the magnitude of the function for this reason:

$$20\log_{10}\left|H(\omega)\right| = 20\log\left|\frac{K(\jmath\omega)^{\pm 1}(1+\frac{\jmath\omega}{z_1})\cdots}{(1+\frac{\jmath\omega}{p_1})\cdots}\right|$$

$$= 20\log|K| + 20\log\left|(\jmath\omega)^{\pm 1}\right| + 20\log\left|1+\frac{\jmath\omega}{z_1}\right| + \cdots - 20\log\left|1+\frac{\jmath\omega}{p_1}\right| - \cdots$$

If we just graph each term individually, we can just add them all together at the end and we will get the semilog plot.

General Process to Plot a Bode Plot

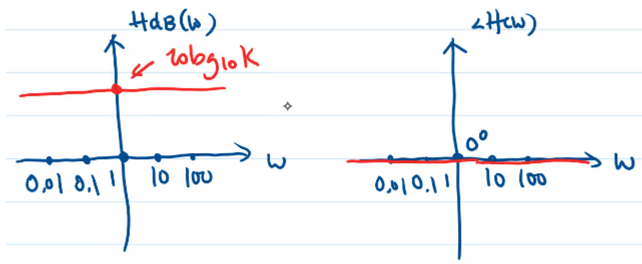
1. Label and mark your axis correctly.
2. Write your transfer function in standard form.
3. Identify all the factors.
4. For each factor, plot the magnitude and phase plot separately. Label the slopes.
5. Add all of the magnitude plots together to get the magnitude plot of your transfer function.
6. Add all of the phase plots together to get the phase plot of your transfer function.

Step 4 is the hardest part since you need to memorize the process for each type of factor. The following is a list of them all.

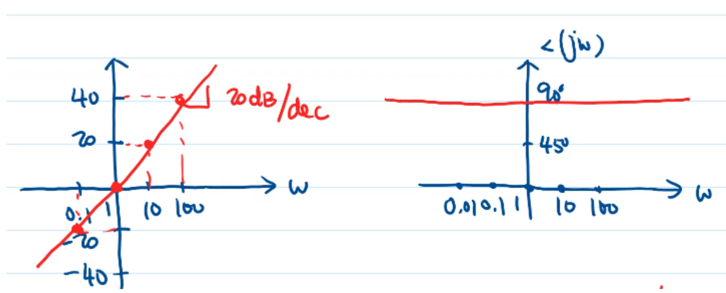
How to Plot the Factors

Always in the order of *magnitude* then *phase*.

1. $H(\omega) = K$

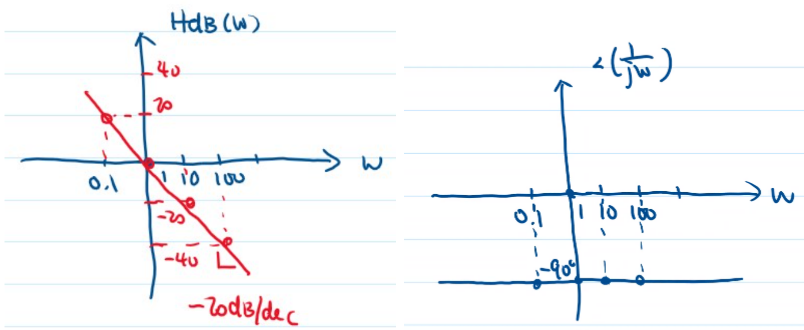


2. $H(\omega) = \jmath\omega$

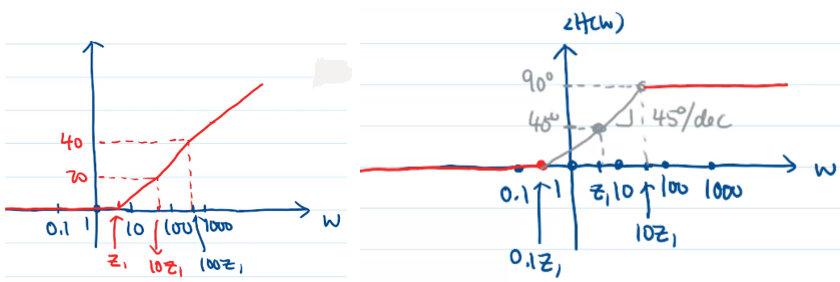


In the previous image, the slope is $20dB/dec$ where dec refers to $decade$ which is a since tick on the $x - axis$, always 10 times bigger than the last.

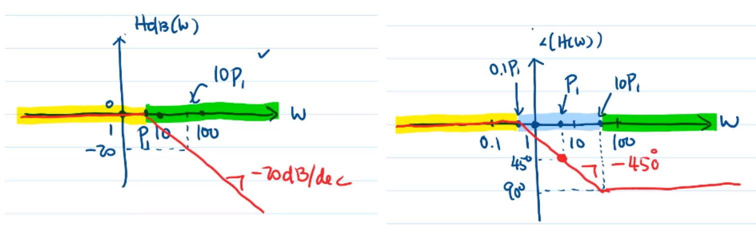
3. $H(\omega) = \frac{1}{\jmath\omega}$



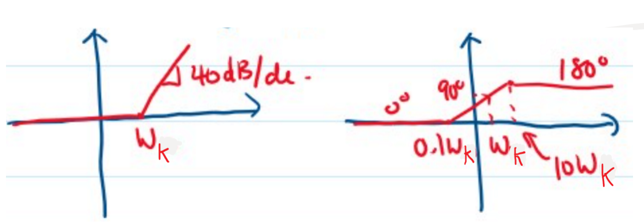
4. **Simple Zero:** $H(\omega) = 1 + \frac{\jmath\omega}{z_1}$



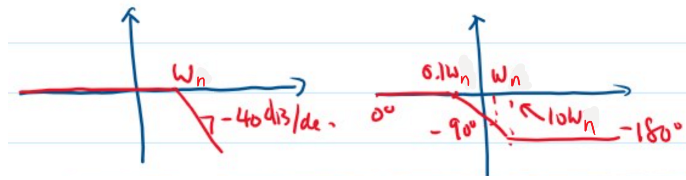
5. **Simple Pole:** $H(\omega) = \frac{1}{1+\frac{\jmath\omega}{p_1}}$



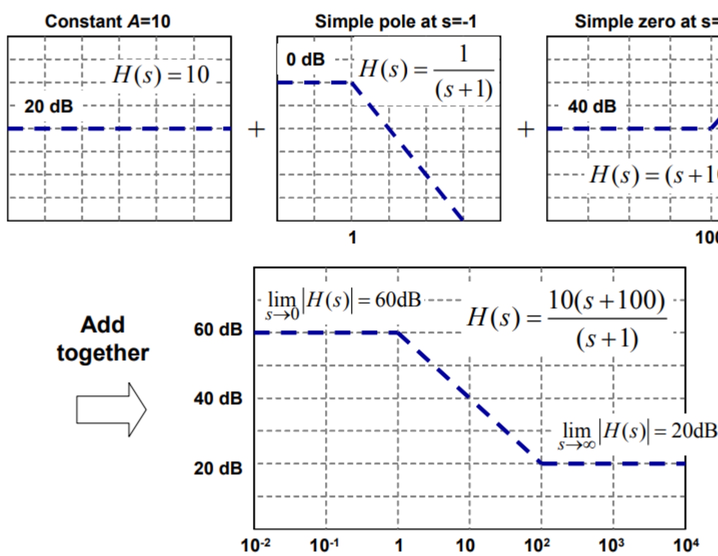
6. **Quadratic Zero:** $H(\omega) = 1 + \frac{\jmath 2\omega}{\omega_k} + \left(\frac{\jmath\omega}{\omega_k}\right)^2$



7. **Quadratic Pole:** $H(\omega) = \frac{1}{1+\frac{j2\omega}{\omega_n}+\left(\frac{j\omega}{\omega_n}\right)^2}$



Once you have plotted the factors individually, you want to combine them like done in this diagram (the image uses s instead of ω):



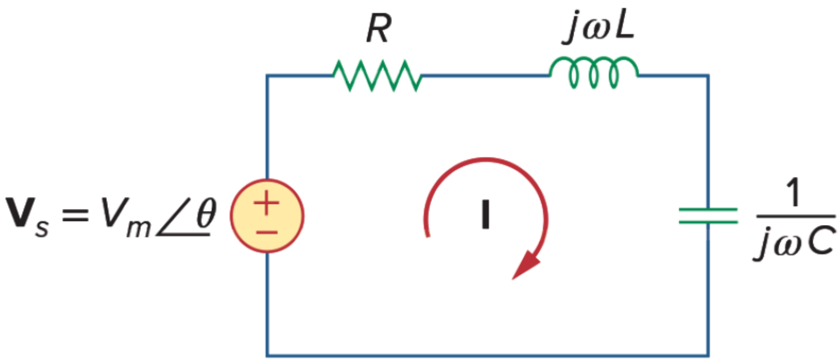
The best way of doing this is by adding up the slopes in all the graphs and denoting intervals. Then connecting it to a point you know at the start of the interval.

14.5: Series Resonance

When evaluating the plots of the magnitude of some transfer functions, there is sometimes a sharp peek in the Bode plot which occurs at some particular frequency. This peek is called the *resonant peek* and the frequency at which it occurs is called the *resonant frequency.*

Resonance is a condition in an RLC circuit in which the impedance is purely resistive, meaning the capacitive and inductive reactants are equal in magnitude.

The following circuit is called the *series resonant circuit*:



If we want to find it's resonant frequency (ω_0), we must calculate at what frequency is the inductive and capacitive impedances equal to each other.

$$\bar{Z} = R + \hat{\jmath}\omega L + \frac{1}{\hat{\jmath}\omega C}$$

$$\bar{Z} = R + \hat{\jmath}\left(\omega L - \frac{1}{\omega C}\right)$$

Now we take just the imaginary part since that is the inductive and capacitive impedance, and set it to 0:

$$\omega_0 L - \frac{1}{\omega_0 C} = 0$$

$$\boxed{\omega_0 = \frac{1}{\sqrt{LC}} rad/s}$$

There are a few things to note here. First of all since the impedance of the circuit is all resistive, this means that the entire voltage drop is across the resistor. Effectively the inductor and capacitors are short circuits at the resonant frequency. This also means that the voltage and current are in phase.

The average power dissipated by the circuit as a function of ω is:

$$P(\omega) = \frac{1}{2} I^2 R$$

At $\omega = \omega_0$, $I = \frac{V_m}{R}$, so:

$$P(\omega_0) = \frac{1}{2} \frac{V_m^2}{R}$$

We are also interested in what frequencies the circuit dissipates half of the maximum power, also called the *half-power frequencies* (ω_1 and ω_2). This means that:

$$P(\omega_1) = P(\omega_2) = \frac{V_m^2}{4R}$$

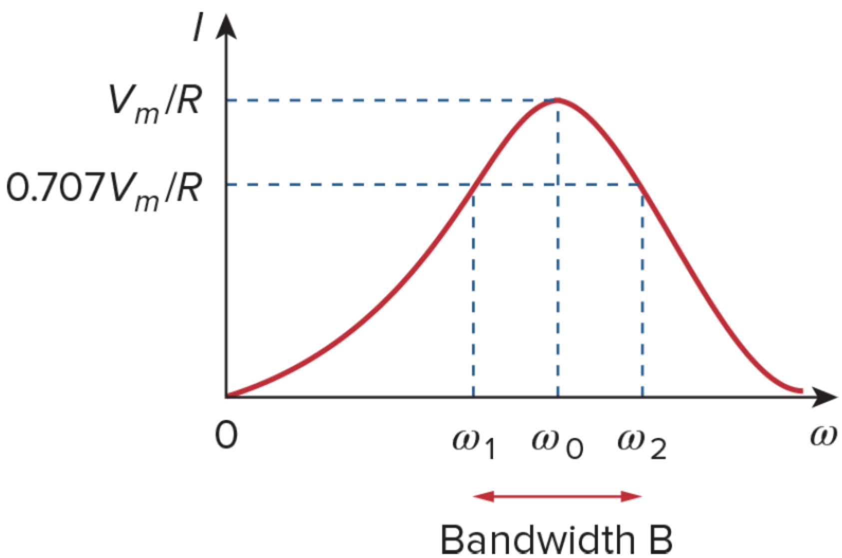
Those frequencies are:

$$\omega_{1,2} = \mp\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

Additionally:

$$w_0 = \sqrt{w_1 w_2}$$

These frequencies have half the power, but around 70.7% of the maximum of the graph, as you can see below. This comes from dividing the maximum value by $\sqrt{2}$.



Bandwidth is defined as the width of the peak:

$$B = \omega_2 - \omega_1$$

The sharpness of the resonant peek is measured using the *quality factor* (Q). At the resonant frequency the reactive energy of the circuit oscillates between the inductor and capacitor, the quality factor relates the maximum energy stored to the energy dissipated in the circuit per cycle of oscillation. In words (for one period):

$$Q = 2\pi\frac{\text{Peak energy stored in the circuit}}{\text{Energy dissipated by the circuit}}$$

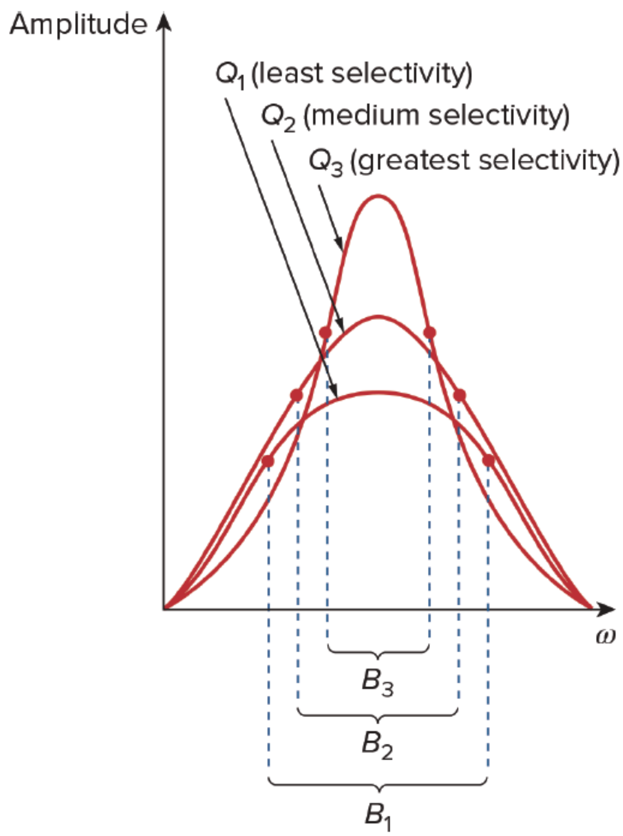
In an equation that is:

$$\boxed{Q = \frac{w_0 L}{R} = \frac{1}{\omega_0 C R}}$$

Furthermore, we can relate the bandwidth and the quality factor:

$$B = \frac{R}{L} = \frac{\omega_0}{Q}$$

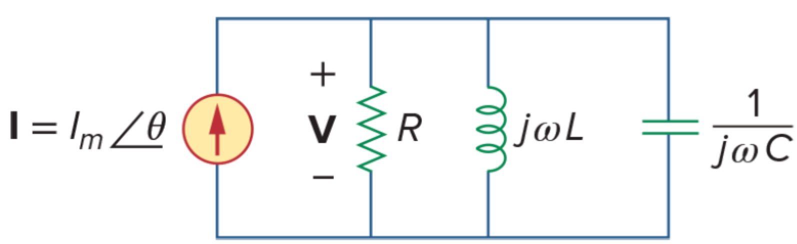
From this we can see if that the quality factor of a frequency response is the ratio of its resonant frequency to its bandwidth. As the quality factor increases the sharpness of the transfer function also increases, becoming more *selective*.



All of the equations mentioned in this section only work for the *series resonant circuit.*

14.6: Parallel Resonance

The following circuit is called the *parallel resonant circuit*:



Using a similar method as before (this time using the admittance) we can conclude:

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

and...

$$\omega_{1,2} = \mp\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}}$$

$$B = \omega_2 - \omega_1 = \frac{1}{RC}$$

$$Q = \frac{\omega_0}{B} = \omega_0 RC = \frac{R}{\omega_0 L}$$

For both series and parallel resonant circuits with a quality factor higher than 10:

$$\omega_{1,2} \approx \omega_0 \mp \frac{B}{2}$$

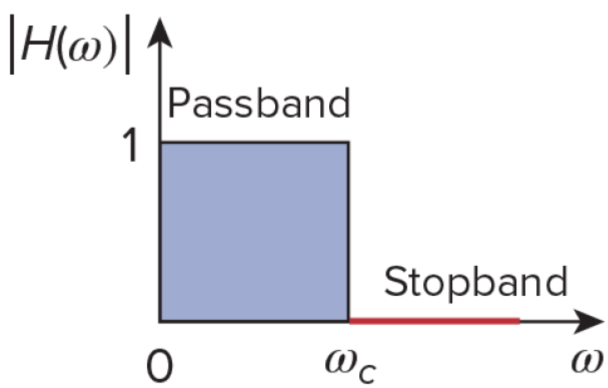
14.7: Passive Filters

A filter is a circuit that is designed to pass signals with desired frequencies and reject others. For example when you listen to the radio, a filter is only allowing signals of a certain frequency to reach your speakers so you only hear one radio station at a time.

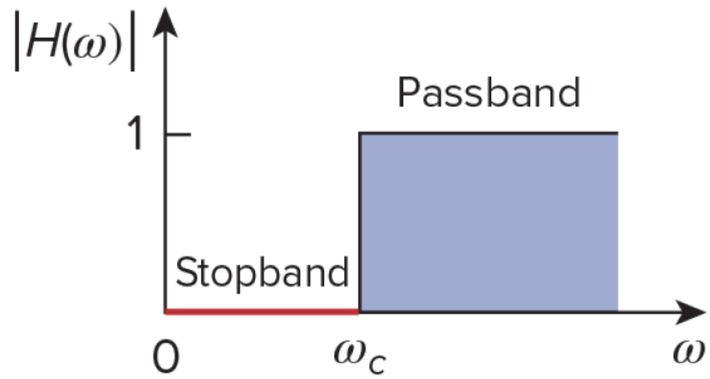
A passive filter is made of RLC elements, as in only passive elements which require no additional energy. An active filter consists of active elements as well as the RLC elements (meaning transistors and op amps).

There are 4 types of filters, regardless of if they are implemented with an active or passive circuit:

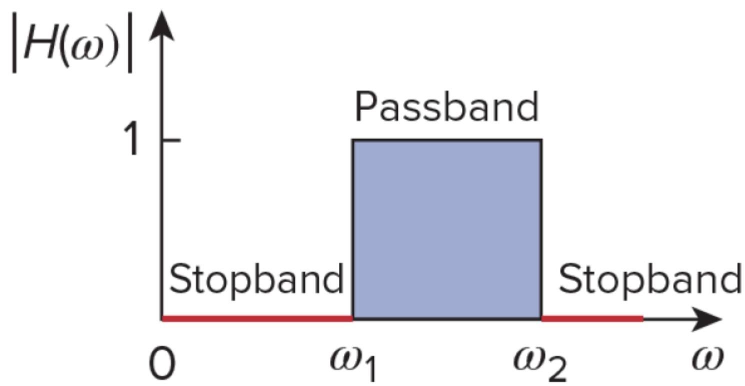
1. **Low-pass Filter** passes low frequencies and stops high frequencies.



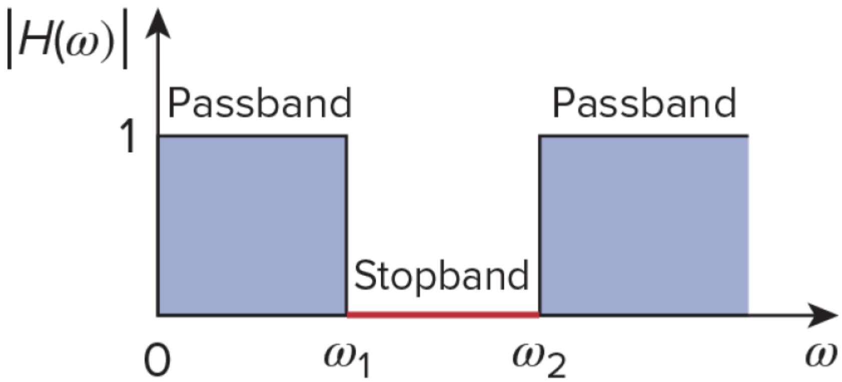
2. **High-pass Filter** passes high frequencies and stops low frequencies.



3. **Band-pass Filter** passes frequencies within a frequency band, and blocks frequencies outside the band.



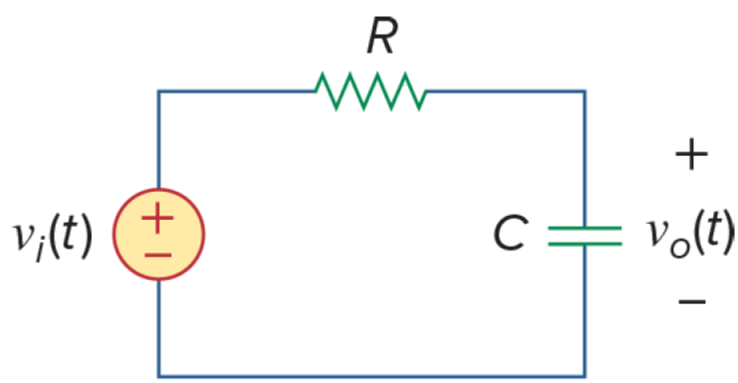
4. **Band-stop Filter** passes frequencies outside a frequency band, and blocks frequencies inside the band.



The previous diagrams are all idealized.

Low-Pass Filters

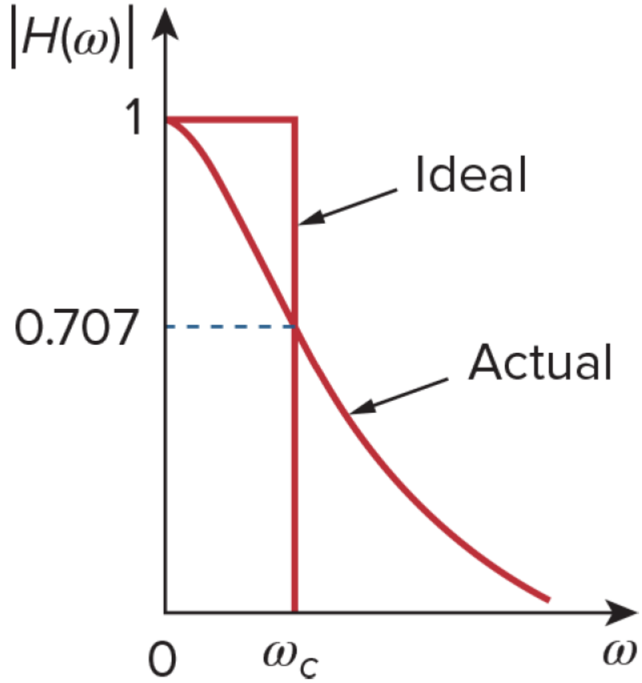
Typically low pass circuits are constructed by taking the voltage off of a RC circuit, like **for example** the following:



This is just an example of a low-pass circuit. We can calculate its transfer function:

$$H(\omega) = \frac{V_o}{V_i} = \frac{1}{1 + \jmath\omega RC}$$

Which gives the following plot. Comparing it to the ideal Low-Pass Filter transfer function you can see how it acts as one.



We are still concerned with the idea of a *half-power frequency* however now in the context of *filters* we call it the *cutoff frequency* or the *rolloff frequency* (ω_c). Once again this will be at $\frac{max}{\sqrt{2}}$ or approximately 70.7% of the maximum of the transfer function.

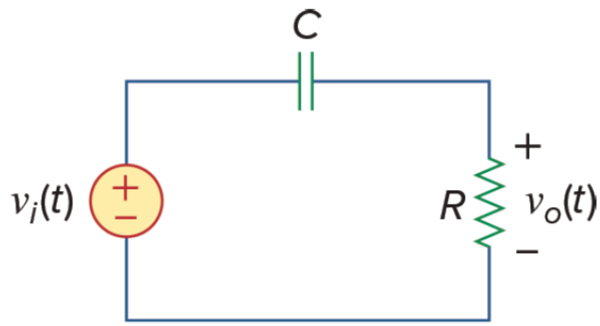
$$\omega_c = \frac{1}{RC}$$

This is where half the power is dissipated in the circuit compared to the maximum.

By definition, a **low-pass filter** is designed to pass signals from dc ($\omega = 0$) to the cutoff frequency ($\omega = \omega_c$).

High-Pass Filter

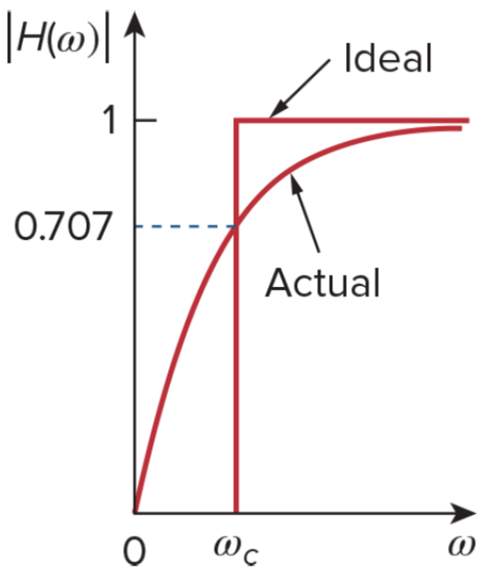
Typically high pass circuits are constructed by taking the voltage off the resistor in an RC circuit. For example the following:



We can then also calculate it's transfer function:

$$H(\omega) = \frac{V_o}{V_i} = \frac{\hat{\jmath}\omega RC}{1 + \hat{\jmath}\omega RC}$$

Which gives the following plot. Comparing it to the ideal high-Pass filter transfer function you can see how it acts as one.

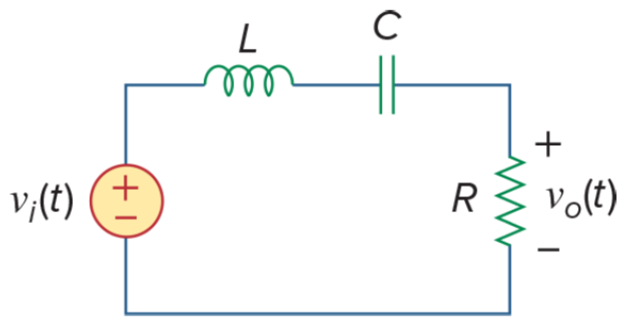


$$\omega_c = \frac{1}{RC}$$

By definition, a **high-pass filter** is designed to stop filters from dc ($\omega = 0$) to the cutoff frequency ($\omega = \omega_c$).

Band-Pass Filters

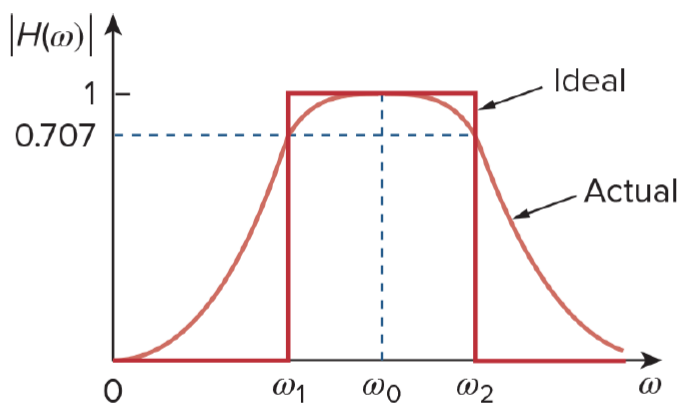
A band-pass filter is typically constructed by taking the voltage off the resistor in the series RLC circuit:



We can calculate the transfer function:

$$H(\omega) = \frac{V_o}{v_i} = \frac{R}{R + \jmath(\omega L - \frac{1}{\omega C})}$$

Which gives the following plot. Comparing it to the ideal band-pass filter transfer function you can see how it acts as one.



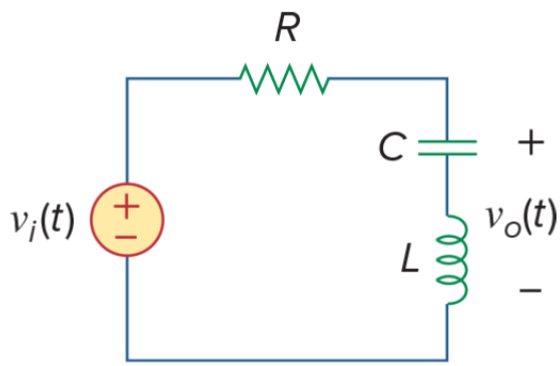
The filters passes signals of a frequency between it's two half power frequencies, centred at ω_0:

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

This filter is a series resonant circuit, and so it follows the equations from that section.

Band-Stop Filters

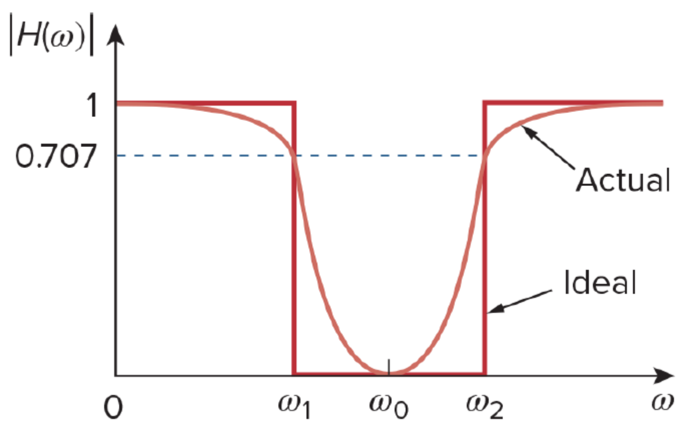
A band-stop filter is typically constructed by taking the voltage off the LC pair in a series RLC circuit:



We can calculate the transfer function:

$$H(\omega) = \frac{V_o}{v_i} = \frac{\hat{\jmath}(\omega L - \frac{1}{\omega C})}{R + \hat{\jmath}(\omega L - \frac{1}{\omega C})}$$

Which gives the following plot. Comparing it to the ideal band-stop filter transfer function you can see how it acts as one.



Once again the stop band is centred at ω_0:

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

In this context, ω_0 is called the frequency of rejection. This filter is a series resonant circuit, and so it follows the equations from that section.

14.8: Active Filters

There some some limitations to passive filters, namely:

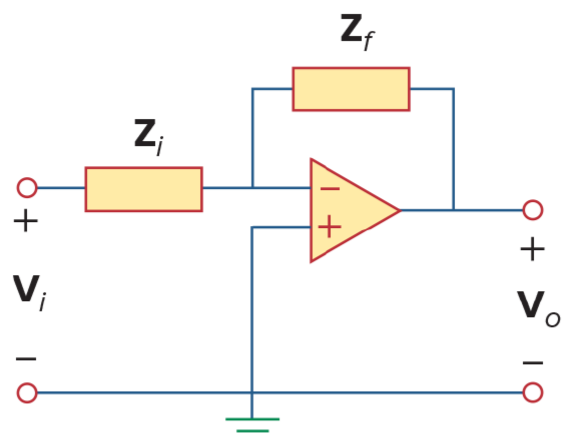
- They can be large due to the inductors.

- They cannot amplify the signal, as in the maximum gain is 1.
- They only function reliably at high frequencies.

Active filters are made of resistors, capacitors, and op amps which fix many of these issues. Active filters require a source of power to operate, and to amplify signals.

First-Order Low-Pass Filter

In general, an active first-order low/high-pass filter is in the form:

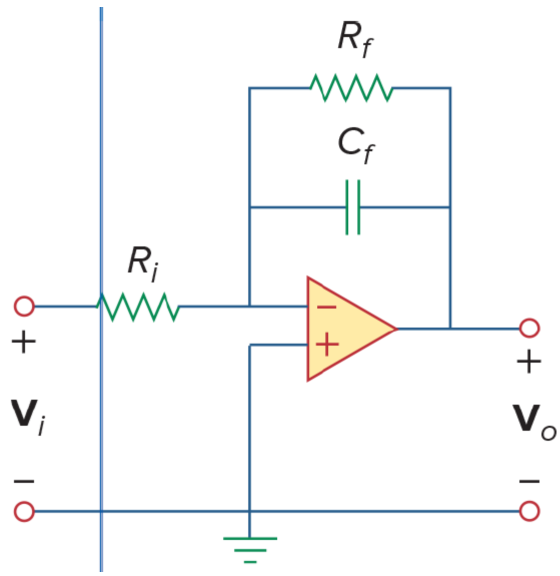


Where components selected for Z_i and Z_f determine whether the filter is low/high-pass, however one of them must be reactive.

The transfer function would be:

$$H(\omega) = \frac{V_o}{V_i} = -\frac{Z_f}{Z_i}$$

A particular example of a low-pass filter could be:



The transfer function is the same as the passive low-pass filter but with a constant gain:

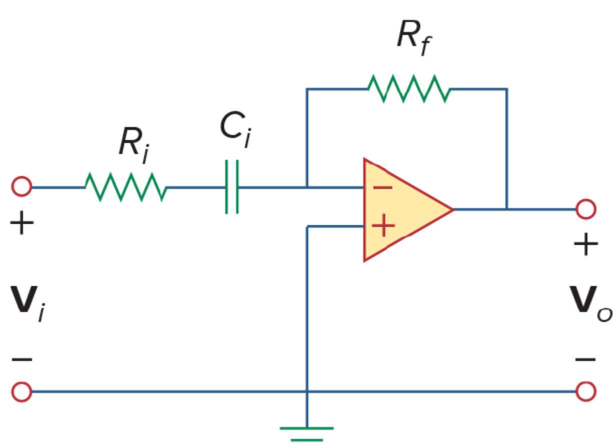
$$H(\omega) = \frac{R_f}{R_i}\frac{1}{1 + \jmath\omega C_f R_f}$$

Where the cutoff frequency is:

$$\omega_c = \frac{1}{R_f C_f}$$

First-Order High-Pass Filter

In the same general configuration as before, the following is an example of an active high-pass filter:



The transfer function is the same as the passive high-pass filter but with a constant gain:

$$H(\omega) = -\frac{\jmath\omega C_i R_f}{1 + \jmath\omega C_i R_i}$$

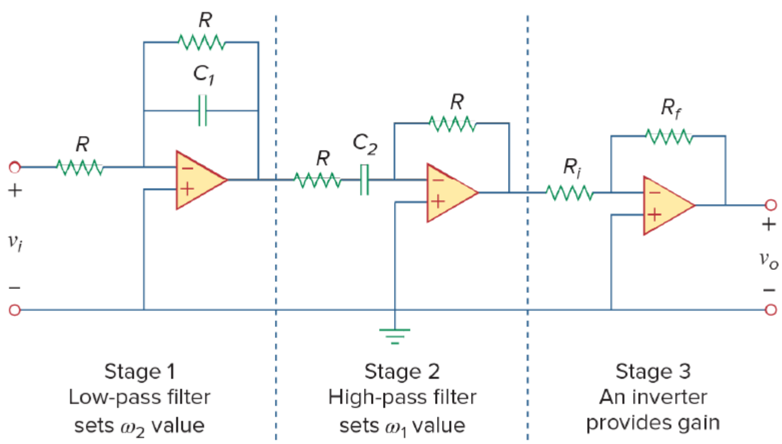
... and the cutoff frequency is:

$$\omega_c = \frac{1}{R_i C_i}$$

Active Band-Pass Filter

By cascading active high pass and low pass filters together, and then inputting the signal into an inverting amplifier, we can achieve an active band-pass filter:

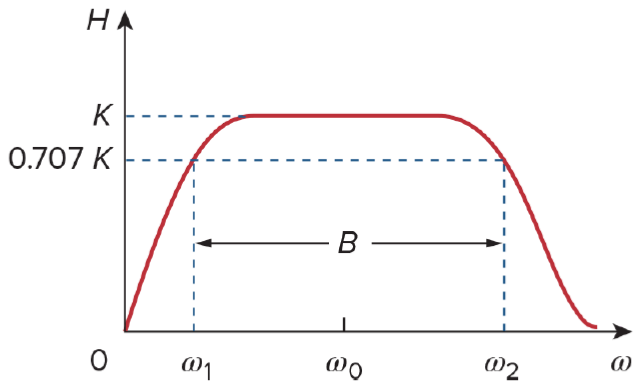
Fully worked with op amps, the circuit looks like the following:



We can calculate the transfer function for this circuit by multiplying together the individual transfer functions:

$$H(\omega) = \frac{V_o}{V_i} = -\frac{R_f}{R_i} \frac{1}{1 + \jmath\omega C_1 R} \frac{\jmath\omega C_2 R}{1 + \jmath\omega C_2 R}$$

A plot of which is:



The low pass filter sets the upper bound ω_2:

$$\omega_2 = \frac{1}{RC_1}$$

The high pass filter sets the lower bound ω_1:

$$\omega_1 = \frac{1}{RC_2}$$

... additionally:

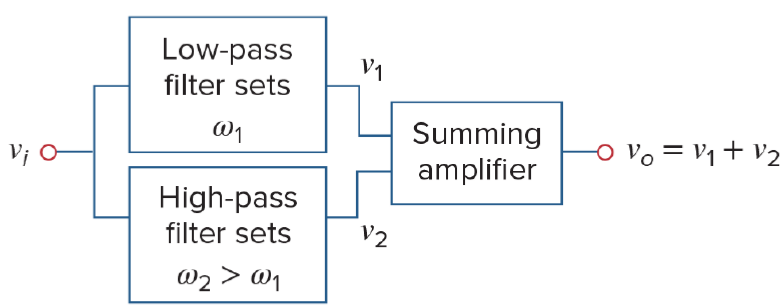
$$w_0 = \sqrt{\omega_1 \omega_2}$$
$$B = \omega_2 - \omega_1$$
$$Q = \frac{\omega_0}{B}$$

We can determine the passband gain (K) to be:

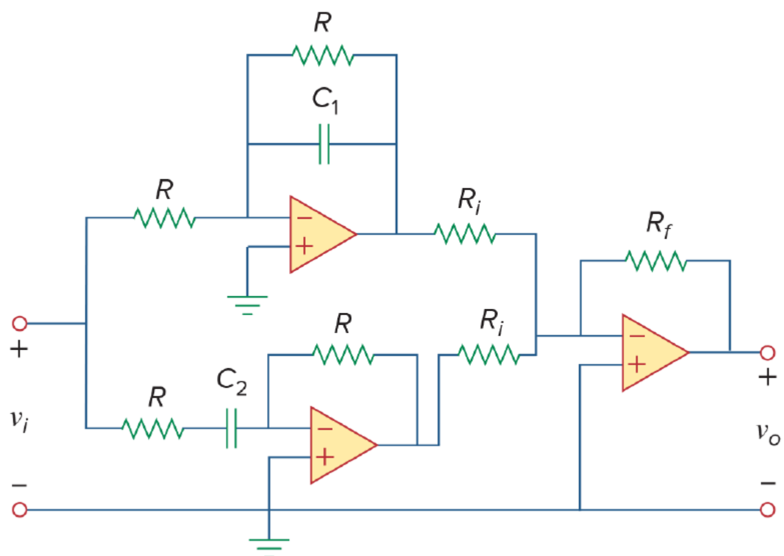
$$K = \frac{R_f}{R_i} \frac{\omega_2}{\omega_1 + \omega_2}$$

Active Band-Reject Filter

By combining two signals passed through filters using a summing amplifier, we can achieve an active band-reject filter:



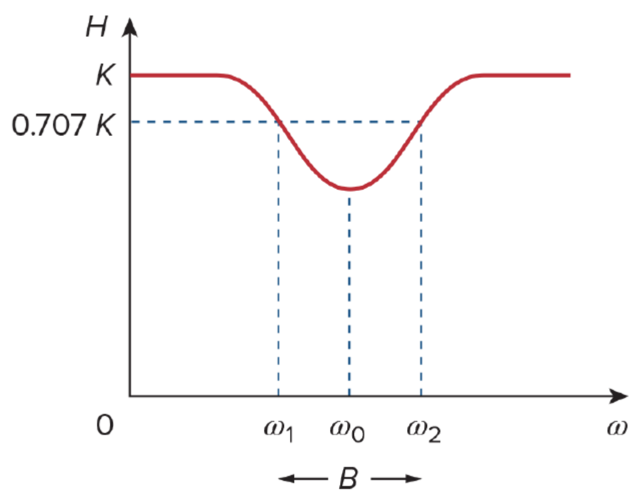
Fully worked with op amps, the circuit looks like the following:



We can calculate the transfer function for this circuit by multiplying together the individual transfer functions:

$$H(\omega) = \frac{V_o}{V_i} = -\frac{R_f}{R_i}\left(-\frac{1}{1+\jmath\omega C_1 R} - \frac{\jmath\omega C_2 R}{1+\jmath\omega C_2 R}\right)$$

A plot of which is:



We can determine the passband gain (K) to be:

$$K = \frac{R_f}{R_i}$$

Chapter 15: Introduction to the Laplace Transform

In AC domain, we used $\jmath\omega$ to replace differentiation. This process however only worked for sinusoidal input functions. We want a method to turn any arbitrary input function into a domain where differentiation is replaced with some simpler operation. For this purpose we use the *Laplace Transform* because of the following property:

$$\mathscr{L}\{f'(t)\} = sF(s) - f(0)$$

... and similar rules apply for higher order derivatives. In a sense, $\frac{d}{dt} \implies s\cdot$ in Laplace Domain. Similarly integration with respect to time is replaced with division by s in Laplace Domain. Laplace Domain is when all the circuit elements are rewritten by applying the Laplace Transform to all their elements.

Some notable features of the edge behaviour is:

$$\text{Initial Value: } f(0) = \lim_{s\to\infty} sF(s)$$

$$\text{Final Value: } f(\infty) = \lim_{s\to 0} sF(s)$$

If we could have an input of 1 in the Laplace domain, then our output would be equal to the transfer function of our circuit, since:

$$H(s) = \frac{O(s)}{I(s)}$$

We then need to ask what function has a Laplace inverse of 1? This is called the *Dirac-Delta Function* or the *Impulse function*.

$$\delta(t-a) = \begin{cases} 0 & t \neq a \\ \infty & t = a \end{cases}$$

The following is true:

$$\mathscr{L}^{-1}\{1\} = \delta(t)$$

If your input to the circuit is $\delta(t)$ then your output is the transfer function. This is impossible in real life because it would imply infinite power because of the discontinuous jump, but it can be approximated.

In an integral, it effectively samples the function it is being multiplied by at the input a, as in:

$$\int_0^\infty \delta(t-a)f(t)dt = f(a)$$

Chapter 16: Applications of the Laplace Transform

We can convert all passive elements to Laplace domain using their $I-V$ characteristics.

Resistors in Laplace Domain

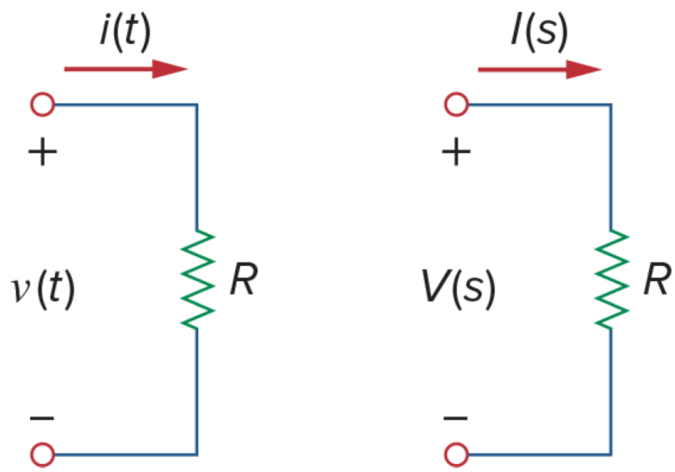
Since:

$$v(t) = Ri(t)$$

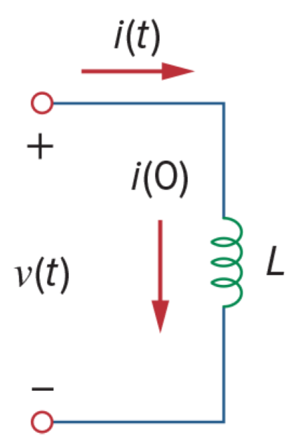
... then:

$$\boxed{V(s) = RI(s)}$$

... once you take the Laplace Transform of both sides:



Inductors in Laplace Domain



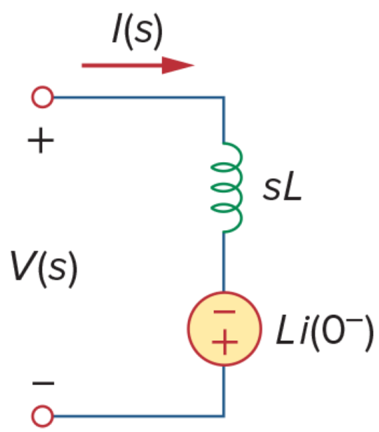
Since:

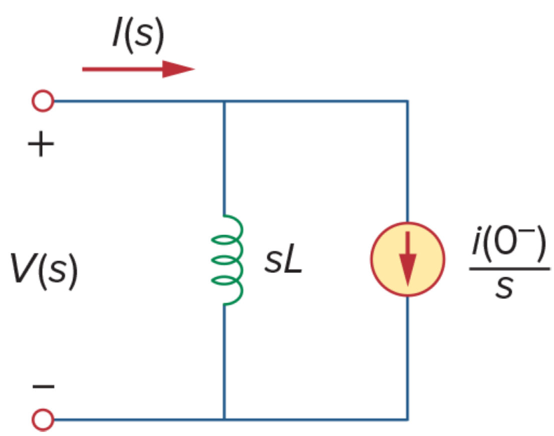
$$v(t) = L\frac{di(t)}{dt}$$

... then:

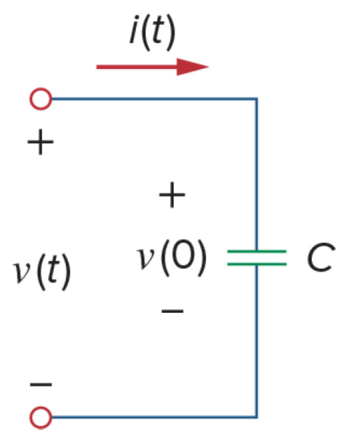
$$\boxed{I(s) = \frac{1}{sL}V(s) + \frac{i(0^-)}{s}}$$

... once you take the Laplace Transform of both sides and rearrange. There are **two** ways to realize this in a circuit:





Capacitors in Laplace Domain



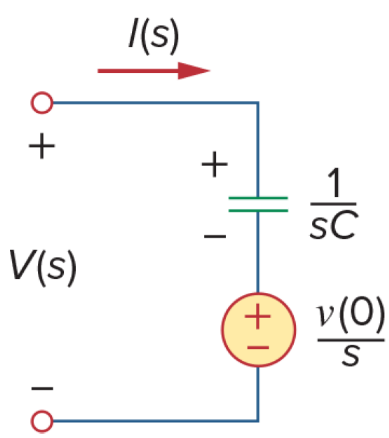
Since:

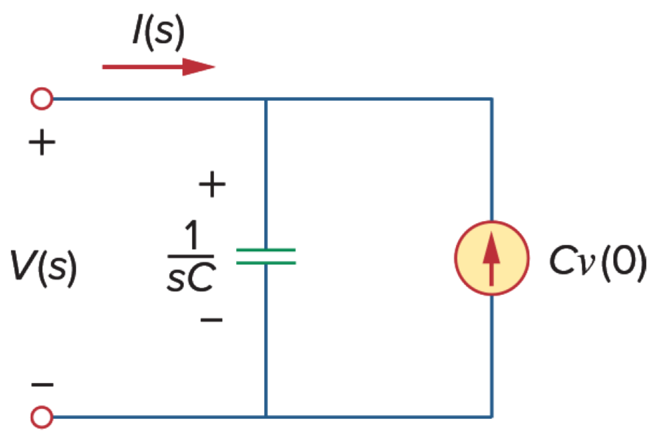
$$i(t) = C\frac{dv(t)}{dt}$$

... then:

$$\boxed{V(s) = \frac{1}{sC}I(s) + \frac{v(0^-)}{s}}$$

... once you take the Laplace Transform of both sides and rearrange. There are **two** ways to realize this in a circuit:





Impedance is defined as usual as the ratio of $V(s)$ to $I(s)$, and so assuming zero initial conditions it is:

$$\text{Resistor: } Z(s) = R$$
$$\text{Inductor: } Z(s) = sL$$
$$\text{Capacitor: } Z(s) = \frac{1}{sC}$$

In Laplace Domain we can use all the same circuit analysis techniques like in DC analysis:

- Node voltage analysis
- Mesh current analysis
- Thevenin/Norton
- Voltage/Current Division

Multiplication in Laplace Domain is the same as convolution in time domain:

$$v(t) = \int_{-\infty}^{\infty} i(\tau)h(t-\tau)d\tau = i(t) * h(t)$$

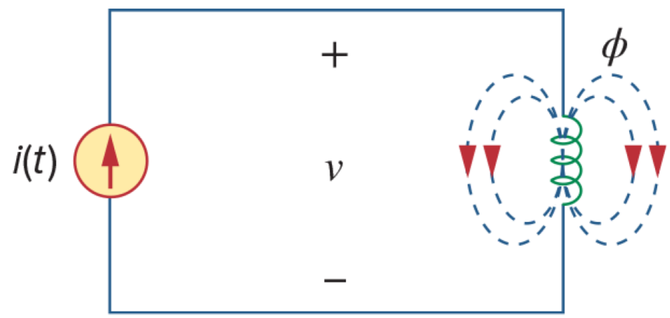
Chapter 13: Magnetically Coupled Circuits

13.2: Mutual Inductance

Circuits are *conductively coupled* if one loop affects neighbouring loop through current conduction. These are circuit couples we have been studying so far. Think about how when doing mesh current analysis, the mesh currents combine to form the current between two loops.

When two loops with or without contacts between them affect each other through the magnetic field generated by one of them,they are said to be magnetically coupled. When two inductors are in close proximity to each other, the magnetic flux caused by current in one coil links with the other coil, inducing a voltage in the latter. This is called mutual inductance.

Given some inductor with some current, the magnetic flux (ϕ) can be through of as:



By Faraday's law:

$$v = N\frac{d\phi}{dt}$$

... where N is the number of coils in the inductor. Note that the flux is caused by a change in current and so:

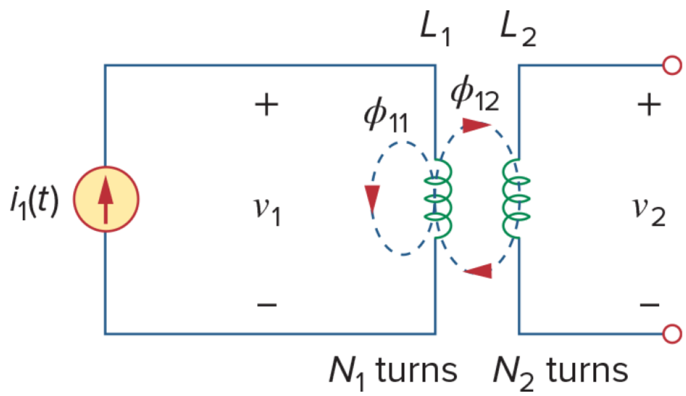
$$v = N\frac{d\phi}{di}\frac{di}{dt}$$

If we define $L = N\frac{d\phi}{di}$, then we get:

$$v = \boxed{N\frac{d\phi}{di}}\frac{di}{dt} \implies v = L\frac{di}{dt}$$

... which is the relationship we already know about the voltage across an inductor. This inductance L is called *self-inductance*.

Now we consider two inductors with self-inductances L_1 and L_2, being in close proximity to each other. Note also that one inductor no current, and so has no magnetic flux.



As you can see, some of the flux of inductor 1 acts only on itself (ϕ_{11} can be thought of as flux from 1 onto 1). Then some of the flux interacts with the other inductor (ϕ_{12}). These fluxes must add to the total flux from the first inductor which is ϕ_1:

$$\phi_1 = \phi_{11} + \phi_{12}$$

Now consider the voltages across either inductor:

$$v_1 = N_1 \frac{d\phi_1}{di_1} \frac{di_1}{dt} = L_1 \frac{di_1}{dt}$$

Similarly:

$$v_2 = N_2 \frac{d\phi_{12}}{di_1} \frac{di_1}{dt}$$

If we define a new type inductance called *mutual inductance* as:

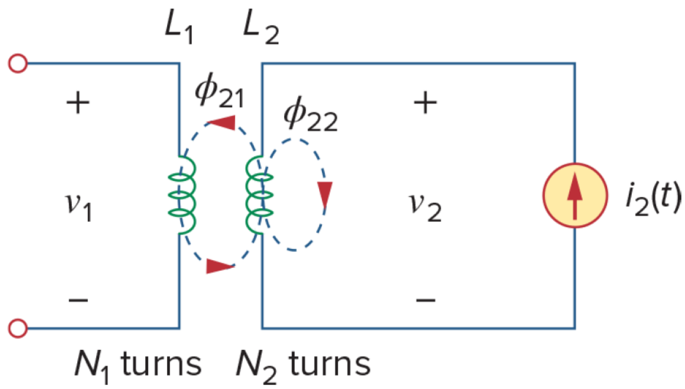
$$M_{21} = N_2 \frac{d\phi_{12}}{di_1}$$

Then:

$$v_2 = M_{21} \frac{di_1}{dt}$$

M_{21} is the mutual inductance of coil 2 with respect to coil 1.

Through the exact same process on the following circuit:



We can conclude that:

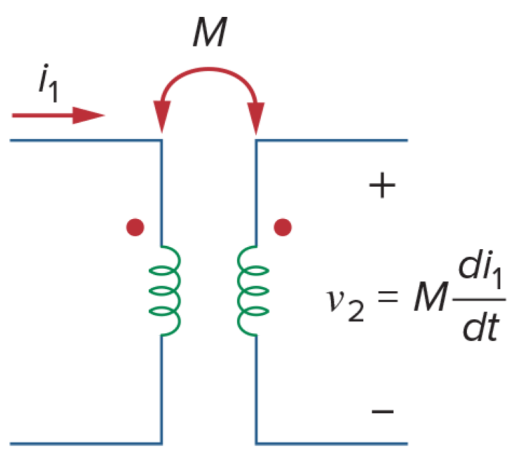
$$v_1 = M_{12}\frac{di_2}{dt}$$

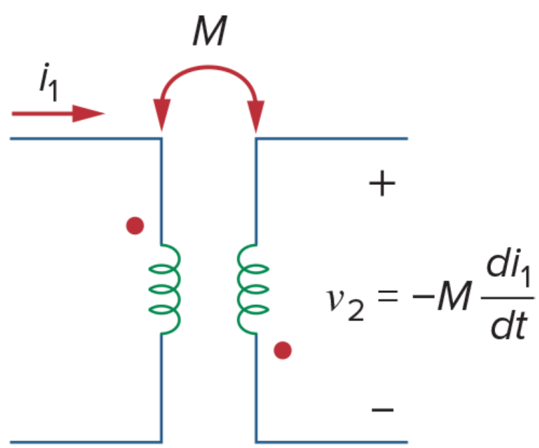
It will be later shown that:

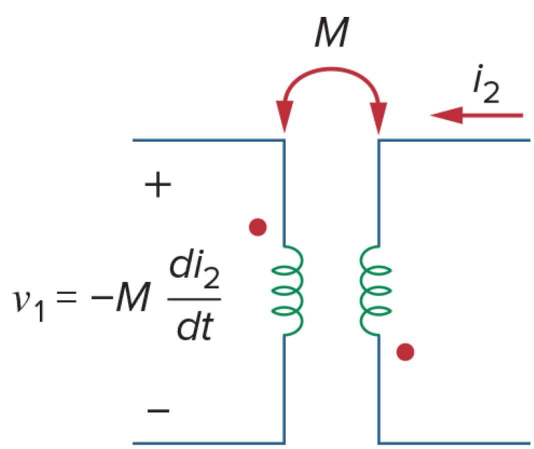
$$M_{21} = M_{12} = M$$

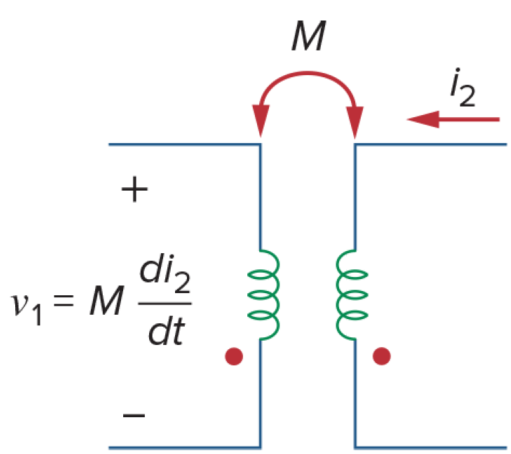
... and we define M as the mutual inductance between two inductors. This is the ability of one inductor to induce a voltage across a neighbouring inductor, measured in Henrys (H). We say that these inductors a *mutually coupled* which can only occur when the inductors are at close proximity, and that the sources are time-varying.

The polarity of the voltage on a single inductor is done by the passive sign convention. The polarity of the voltages across two magnetically coupled circuits is dependent on the orientation and physical winding of each inductor. The dot convention is used to show the polarities. If current enters the dotted terminal of one coil, the reference polarity of the mutual voltage in the second coil is positive at the dotted terminal, visually that is:



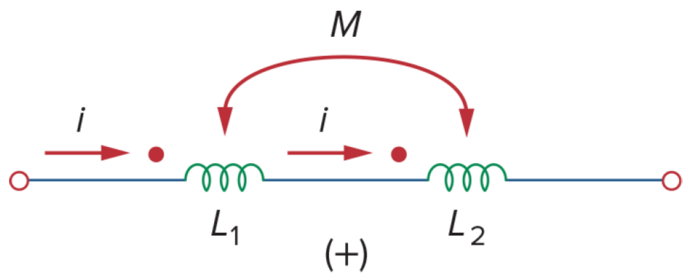






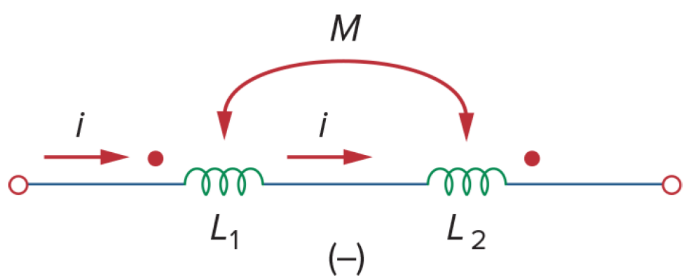
Inductors in series can also be mutually coupled, and can then be replaced by just a single equivalent inductor, as follows:

Series-Aiding Connection



$$L_{eq} = L_1 + L_2 + 2M$$

Series-Opposing Connection



$$L_{eq} = L_1 + L_2 - 2M$$

13.3: Energy in a Coupled Circuit

The mutual inductance cannot be greater than the geometric mean of the self-inductances of the coils. The extent to which the mutual inductance approaches the upper limit is called the *coefficient of coupling* k given by:

$$k = \frac{M}{\sqrt{L_1 L_2}}$$

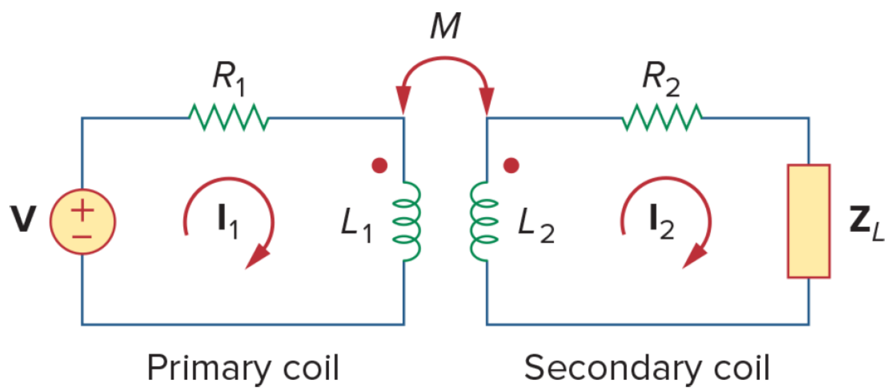
Since mutual inductance cannot be negative, $0 \leq k \leq 1$.

If the entire flux produced by one coil links another coil, then $k = 1$ and we have 100% coupling and they are said to be *perfectly coupled*.

In general the coupling coefficient k is a measure of the magnetic coupling between two coils. This logically depends on the distance between the two coils, their core, their orientation, and their windings.

13.4: Linear Transformers

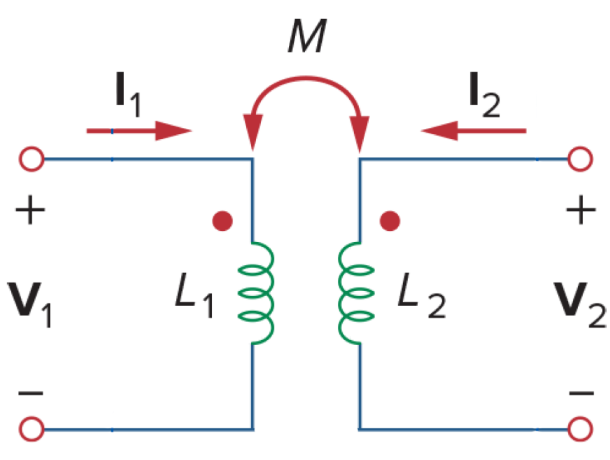
A transformer is a circuit element which takes advantage of mutual inductance. Generally it is a four terminal device comprising two or more magnetically coupled coils. The following is a general schematic of a linear inductor:



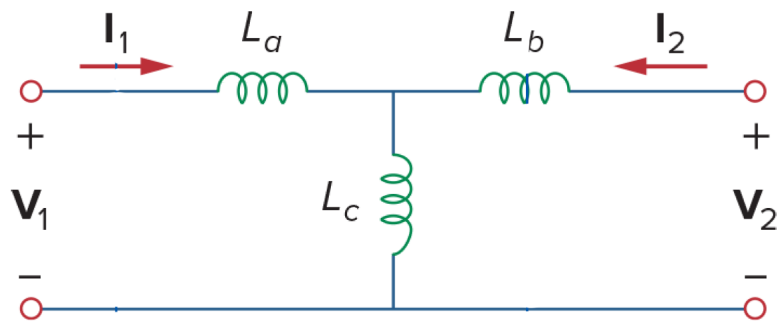
The coil with the voltage source is said to be *primary* while the coil with the load is said to be *secondary*. The resistances are included to account for the losses in the coils. The transformer is said to be linear if the coils are wound on magnetically linear material, meaning the magnetic permeability is constant. Most materials are magnetically linear. In other words, flux is proportional to the current in its windings.

The function of transformers is described in the next section, for now we would like to be able to transform a linear transformer into one of two configurations.

We want to transform the following initial circuit, and values:



The first configuration is called the T circuit:



Where:

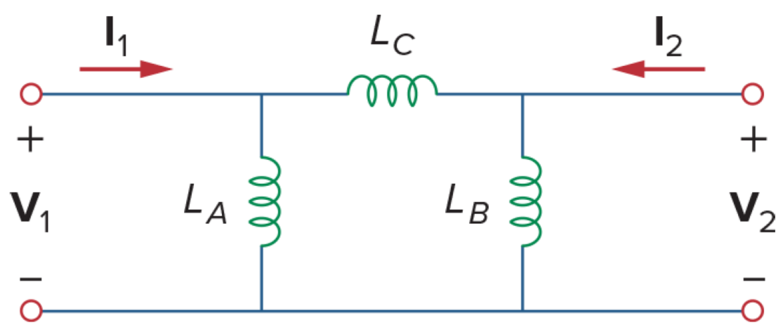
$$L_a = L_1 - M$$

$$L_b = L_2 - M$$

$$L_c = M$$

... or if the dots are on opposite ends use $-M$.

The second configuration is called the $\prod$ circuit:



Where:

$$L_a = \frac{L_1 L_2 - M^2}{L_2 - M}$$

$$L_b = \frac{L_1 L_2 - M^2}{L_1 - M}$$

$$L_c = \frac{L_1 L_2 - M^2}{M}$$

... or if the dots are on opposite ends use $-M$.

13.5: Ideal Transformers

An ideal transformer is one which:

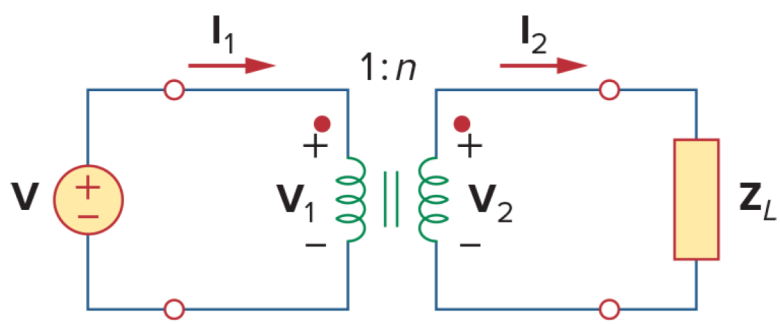
1. Coils have very large reactances ($L_1, L_2, M \to \infty$).

2. Coupling coefficient is equal to unity ($k = 1$), this implies

$$M = \sqrt{L_1 L_2}$$

3. Primary and secondary coils are lossless ($R_1 = R_2 = 0$).

Given some ideal transformer in the following circuit (note the two vertical lines which indicate it is ideal):



We define n to be the turns ratio of the coils, as in:

$$n = \frac{N_2}{N_1}$$

... where N_a is the number of coils in inductor a.

We can further derive:

$$\boxed{n = \frac{N_2}{N_1} = \frac{V_2}{V_1} = \frac{I_1}{I_2}}$$

Notice current and voltage are inversely proportional to conserve energy. There are two simple rules to decide the sign of n:

1. If V_1 and V_2 are *both* positive or both negative at the dotted terminals, use $+n$. Otherwise use $-n$.

2. If I_1 and I_2 *both* enter into or both leave the dotted terminals, use $+n$. Otherwise use $-n$.

- If $n = 1$ we call it an *isolation transformer*.

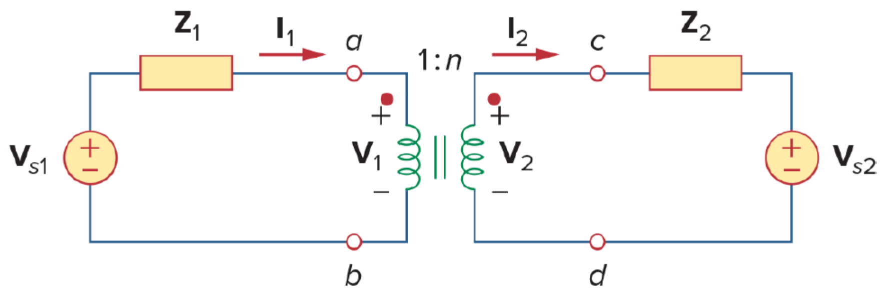
- If $n > 1$ we have a *step-up transformer* since voltage increases from primary to secondary loops. Therefore current is decreased in the second loop.

- If $n < 1$ we have a *step-down transformer* since voltage decreases from primary to secondary loops. Therefore current is increased in the second loop.

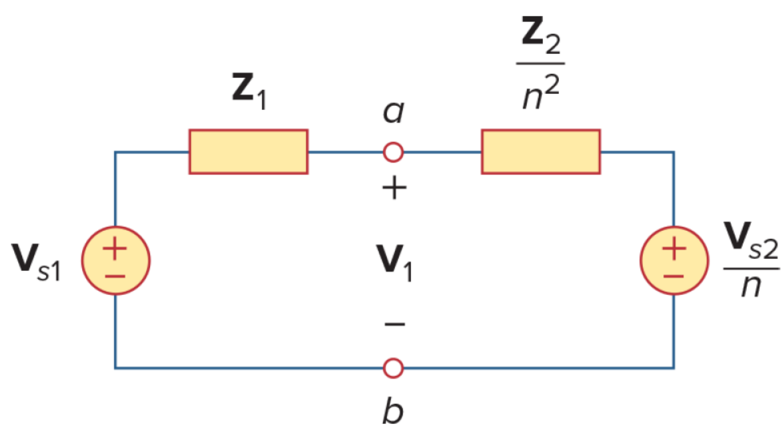
Importantly the **equivalent impedance** (Z_{eq}) felt by the source $\bar{V}$ in the circuit above is:

$$Z_{eq} = \frac{Z_L}{n^2}$$

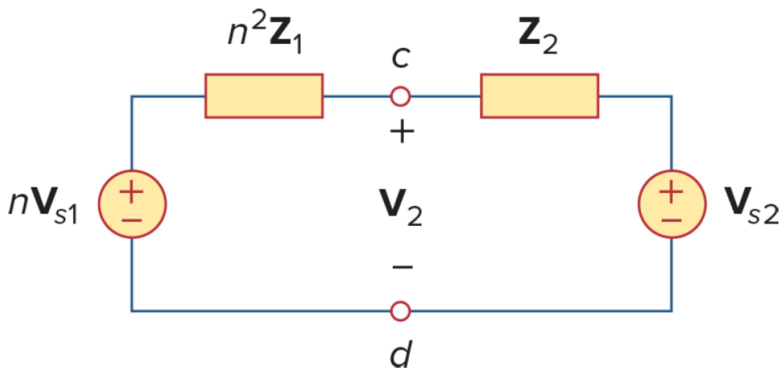
Finally, we look at how to transform the following ideal transformer circuit (which contains sources on either side):



We can either transform the circuit into an equivalent circuit from the perspective of $\bar{V}_{s1}$:



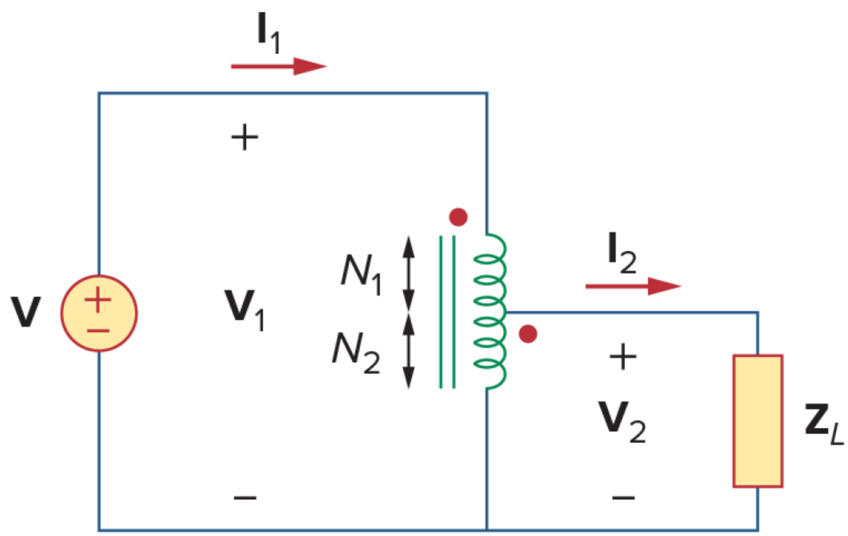
Or we can transform the circuit in to an equivalent circuit from the perspective of $\bar{V}_{s2}$:



13.6: Ideal Autotransformers

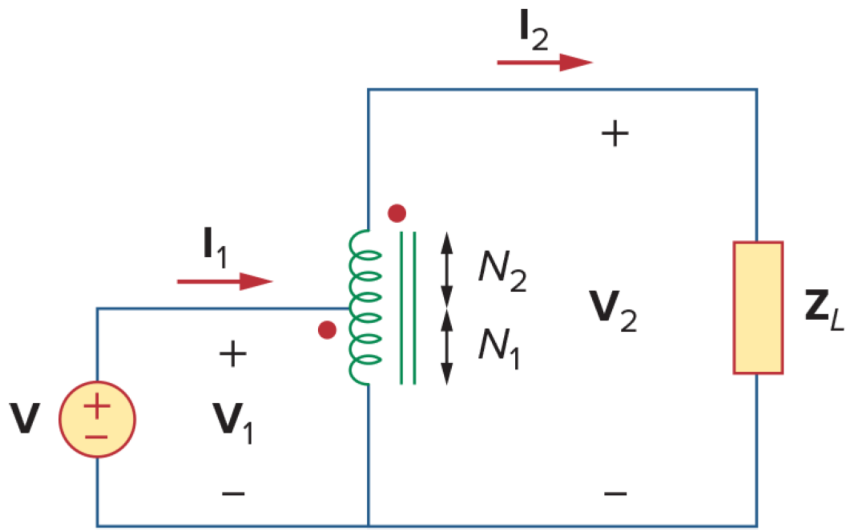
Autotransformers are made of a single continuous winding with a connection point called a *tap* between the primary and secondary sides.

Step-down Autotransformer



$$\frac{\bar{V}_1}{\bar{V}_2} = \frac{N_1 + N_2}{N_1} = 1 + \frac{N_1}{N_2}$$

Step-up Autotransformer



$$\frac{\bar{V}_1}{\bar{V}_2} = \frac{N_1}{N_1 + N_2}$$

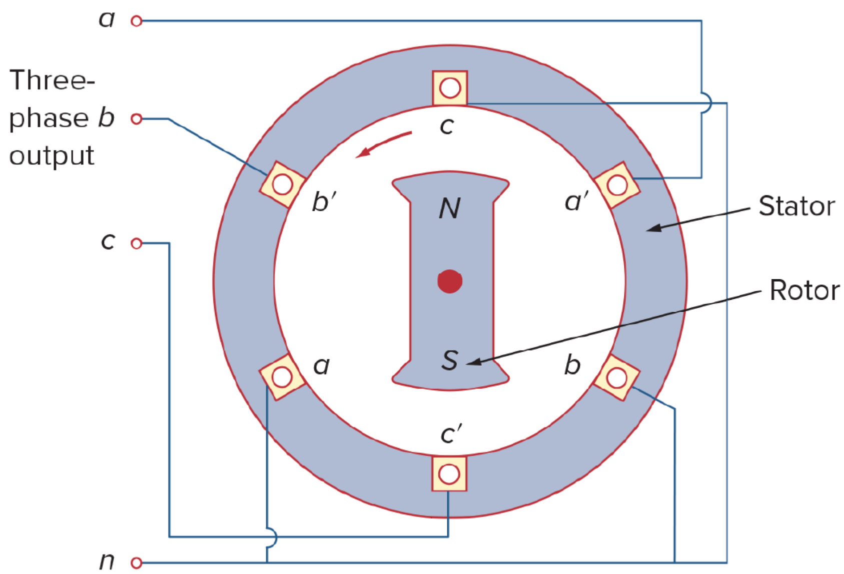
The main difference between an transformer and an autotransformer is that the inductors are both conductively and magnetically coupled. Electrical isolation is not possible with an autotransformer.

Chapter 12: Three-Phase Circuits

Balanced Three-Phase Elements

So far we have studied single-phase circuits. These are circuits in which all the sources are in phase. A *polyphase* circuit is one in which sources are out of phase. A two-phase circuit has two sources which are out of phase from each other by $-90°$. A three-phase circuit has three sources, all $120°$ out of phase of each other. This type of circuit is very useful out of the *polyphase* circuits, and so it is the focus of this chapter.

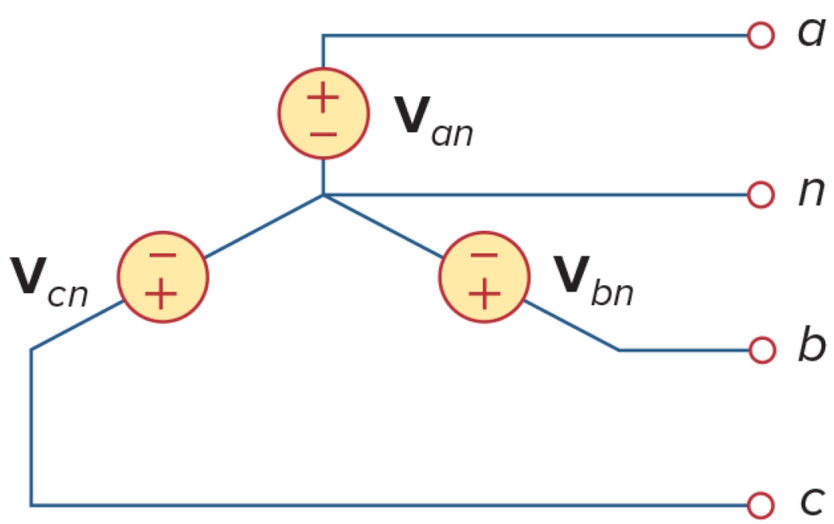
Three phase circuits are created by a three phase generator such as the following:



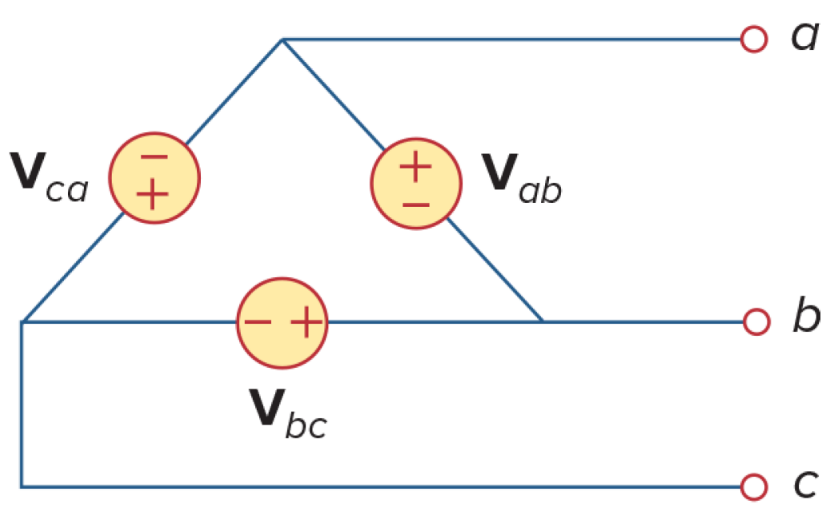
A spinning magnet called the rotor induces an AC voltage with frequency ω on each inductor a, b, c (a', b', c' are their respective grounds). The voltages a, b, c are out of phase by 120° due to their geometric position around the circle.

There are two possible implementations of this generator in a circuit.

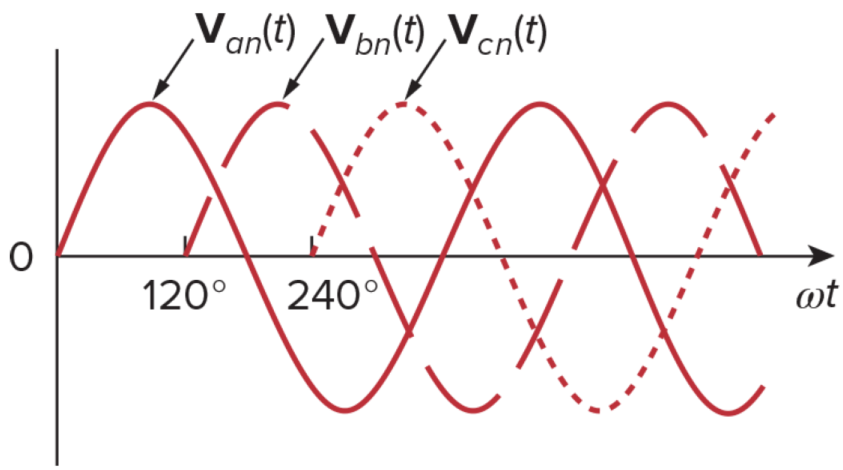
1. $Y-$connected source:



2. $\Delta-$connected source:



The following is a general plot of the voltages in relation to each other:



There are two possible sequences of peak voltages (also called the phase sequences), either:

1. *Positive Sequence*:

$$a \to b \to c \to a \to b \to c \to a \to \dots$$

... in this case:

$$V_{an} = V_p \angle 0°$$
$$V_{bn} = V_p \angle -120°$$
$$V_{cn} = V_p \angle 120° (= V_p \angle -240°)$$

2. *Negative Sequence*:

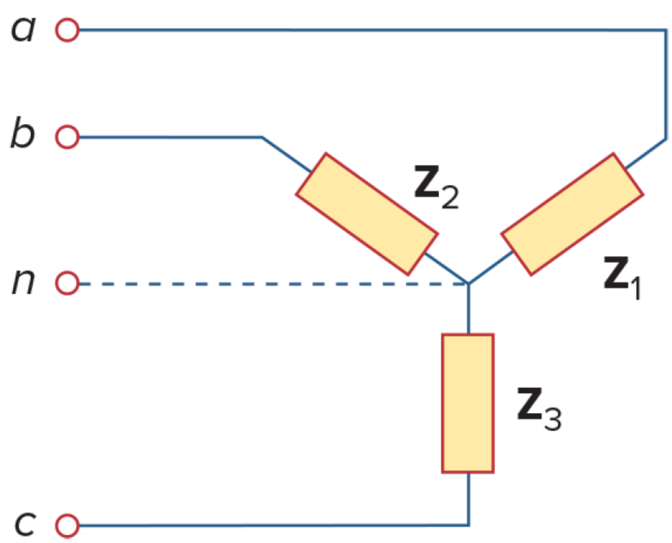
$$a \to c \to b \to a \to c \to b \to a \to \dots$$

... in this case:

$$V_{an} = V_p \angle 0°$$
$$V_{bn} = V_p \angle 120° (= V_p \angle -240°)$$
$$V_{cn} = V_p \angle -120°$$

Loads can also be in three-phase form, and just like sources, the can be either in Y or Δ configuration.

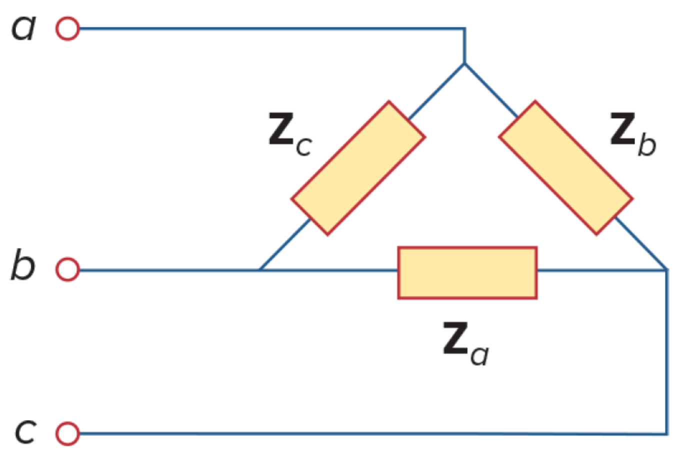
1. $Y-$connected three-phase load.



This type of load is said to be in *balanced* if:

$$Z_1 = Z_2 = Z_3 = Z_Y$$

2. $\Delta-$connected three-phase load.



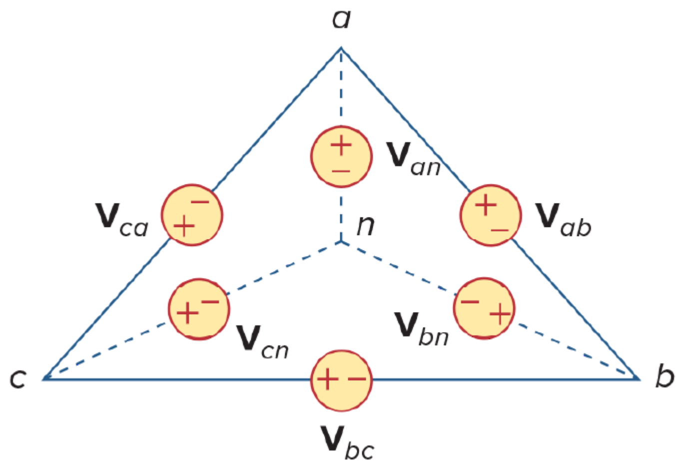
This type of load is said to be in *balanced* if:

$$Z_a = Z_b = Z_c = Z_\Delta$$

You can use the following equation to convert a balanced three-phase load between forms:

$$Z_\Delta = 3Z_Y$$

You can also convert between Δ and Y sources as in the following image and equations:



$$V_{an} = \frac{V_p}{\sqrt{3}}\angle - 30^\circ$$

$$V_{bn} = \frac{V_p}{\sqrt{3}}\angle - 150^\circ$$

$$V_{cn} = \frac{V_p}{\sqrt{3}}\angle 90^\circ$$

There are thus four possible combinations of three-phase sources and three-phase loads (source then load):

1. $Y - Y$ connection.
2. $Y - \Delta$ connection.
3. $\Delta - \Delta$ connection.
4. $\Delta - Y$ connection.

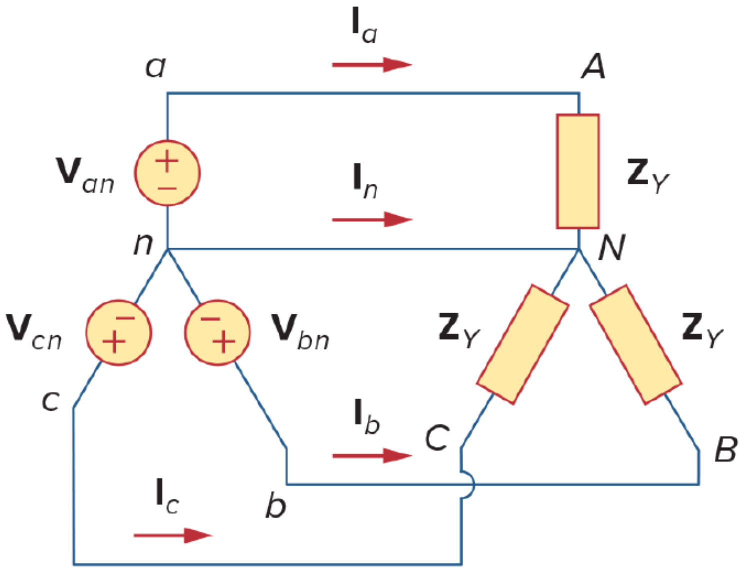
The following four sections of this chapter look at each of these connections in detail.

In the following sections we define the *phase voltages/current* as well at the *line voltages/currents* for all the different configurations. The *phase voltages* are the voltages between the lines a, b, c and n in the Y configuration. The *line voltages* refer to the voltage differences between the three lines. *Phase current* is the current in each phase of the source or load. *Line current* is the current in each transmission line.

In general, in a balanced three-phase circuit, the neutral line can be ignored since the voltage across it is 0 and the current through it is 0.

12.3: Balanced Y-Y Connection

The following is a general $Y - Y$ three phase circuit:



The following equations relate to this circuit:

Phase Voltages

$$V_{an} = V_p \angle 0°$$

$$V_{bn} = V_p \angle -120°$$

$$V_{cn} = V_p \angle 120°$$

Line Voltages

$$V_{ab} = \sqrt{3} V_p \angle 30°$$

$$V_{bc} = V_{ab} \angle -120°$$

$$V_{ca} = V_{ab} \angle 120°$$

Phase Currents

Same as line currents.

Line Currents

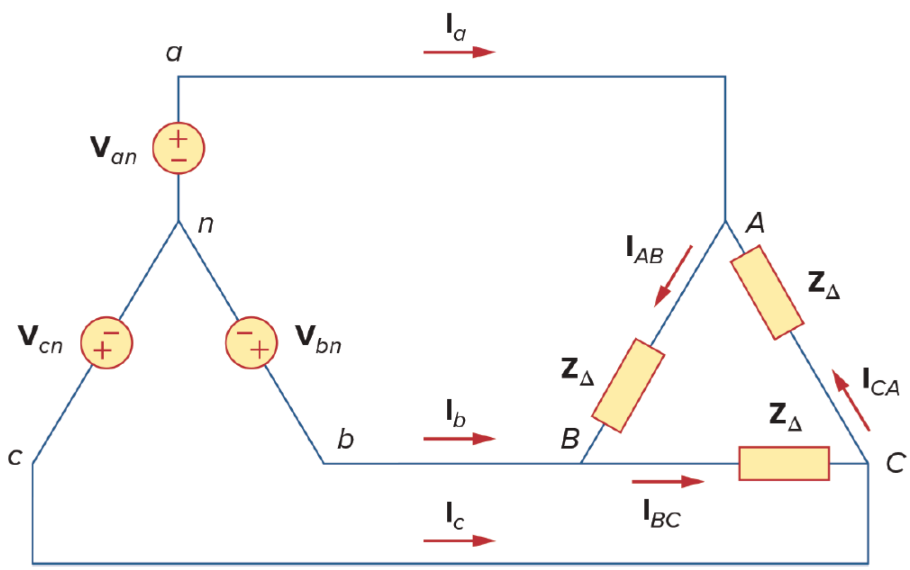
$$I_a = \frac{V_{an}}{Z_Y}$$

$$I_b = I_a \angle -120°$$

$$I_c = I_a \angle 120°$$

12.4: Balanced Y-D Connection

The following is a general $Y - D$ three phase circuit:



The following equations relate to this circuit:

Phase Voltages

$$V_{an} = V_p\angle 0°$$

$$V_{bn} = V_p\angle - 120°$$

$$V_{cn} = V_p\angle 120°$$

Line Voltages

$$V_{ab} = V_{AB} = \sqrt{3}V_p\angle 30°$$

$$V_{bc} = V_{BC} = V_{ab}\angle - 120°$$

$$V_{ca} = V_{CA} = V_{ab}\angle 120°$$

Phase Currents

$$I_{AB} = \frac{V_{AB}}{Z_\Delta}$$

$$I_{BC} = \frac{V_{BC}}{Z_\Delta}$$

$$I_{CA} = \frac{V_{CA}}{Z_\Delta}$$

Line Currents

$$I_a = I_{AB}\sqrt{3}\angle - 30°$$

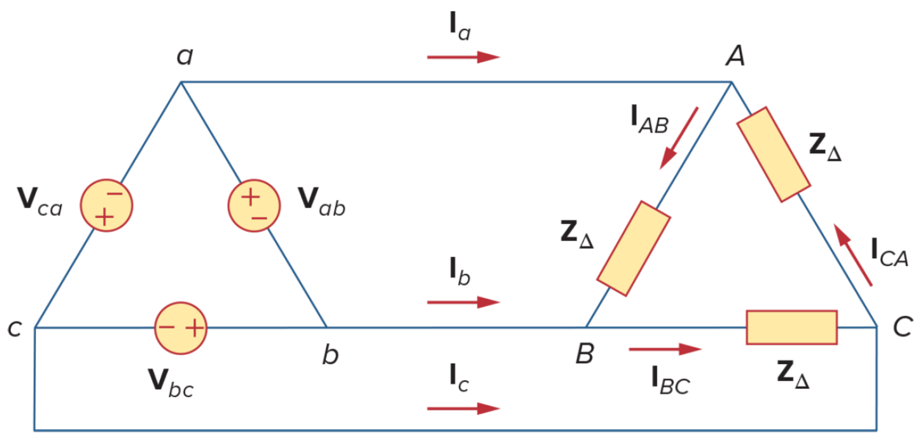
$$I_b = I_a \angle -120°$$

$$I_c = I_a \angle 120°$$

You can also solve these kinds of circuits by transforming the load into a Y topology.

12.5: Balanced D-D Connection

The following is a general $D-D$ three phase circuit:



The following equations relate to this circuit:

Phase Voltages

$$V_{ab} = V_p \angle 0°$$

$$V_{bc} = V_p \angle -120°$$

$$V_{ca} = V_p \angle 120°$$

Line Voltages

Same as phase voltages.

Phase Currents

$$I_{AB} = \frac{V_{ab}}{Z_\Delta}$$

$$I_{BC} = \frac{V_{bc}}{Z_\Delta}$$

$$I_{CA} = \frac{V_{ca}}{Z_\Delta}$$

Line Currents

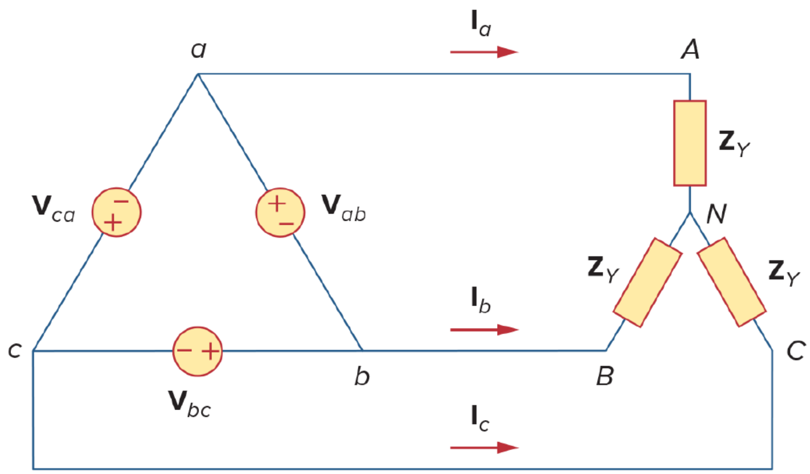
$$I_a = I_{AB}\sqrt{3}\angle -30°$$

$$I_b = I_a \angle -120°$$

$$I_c = I_a \angle 120°$$

12.6: Balanced D-Y Connection

The following is a general $D - Y$ three phase circuit:



The following equations relate to this circuit:
Phase Voltages

$$V_{ab} = V_p\angle 0°$$

$$V_{bc} = V_p\angle -120°$$

$$V_{ca} = V_p\angle 120°$$

Line Voltages

Same as phase voltages.

Phase Currents

Same as line currents.

Line Currents

$$I_a = \frac{V_p\angle -30°}{\sqrt{3}Z_Y}$$

$$I_b = I_a\angle -120°$$

$$I_c = I_a\angle 120°$$

12.7 Power in a Balanced System

This section is essentially just a bunch of formulas relating to power in a three phase circuit.

To begin we look at the total instantaneous power absorbed by the load which is the sum of the instantaneous powers in the three phases:

$$p = 3V_pI_p\cos(\theta)$$

... where θ is the lag of the current behind corresponding phase voltages.

This is a time independent quantity, and it true regardless of the topology of the load.

The following are other formulas:

1. Average power per phase:
$$P_p = V_p I_p \cos(\theta)$$
2. Total average power:
$$P = 3V_p I_p \cos(\theta) = \sqrt{3} V_L I_L \cos(\theta)$$
3. Reactive power per phase:
$$Q_p = V_p I_p \sin(\theta)$$
4. Total reactive power:
$$Q = 3V_p I_p \sin(\theta) = 3Q_p = \sqrt{3} V_L I_L \sin(\theta)$$
5. Apparent power per phase:
$$S_p = V_p I_p$$
6. Complex power per phase:
$$S_p = P_p + \jmath Q_p = V_p I_p^*$$
7. Total complex power:
$$S = 3S_p = 3V_p I_p^* = 3I_p^2 Z_p = \frac{3V_p^2}{Z_p^*} = P + \jmath Q = \sqrt{3} V_L I_L \angle \theta$$
... where Z_p is the load impedance per phase.

Chapter 17: The Fourier Series

The Fourier Series is a method to decompose a periodic function into the sum of a constant term and infinitely many sinusoidal terms. This is useful in our study of circuits, because a periodic source can be then decomposed into a DC source, and then infinitely many phasors. For applications we can just use the first n phasors to get an approximation.

In general these are the steps to solve a a Fourier Series question:

1. Use Fourier series to decompose any periodic function into a DC component and many AC components.

2. Solve using AC and DC techniques separately by "turning off" one source at a time.

3. Use the superposition principle of a linear system to add all the results back together.

All of the theory and methods to compute the Fourier Series of a function is in my *MTH312* notes. We do it only slightly differently here:

To compute the Fourier series $f(t)$ of a function:

$$f(t) = a_0 + \sum_{n=1}^{\infty} \big(a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)\big)$$

... where:

$$\omega_0 = \frac{2\pi}{T}$$
$$a_0 = \frac{1}{T}\int_0^T f(t)dt$$
$$a_n = \frac{2}{T}\int_0^T f(t)\cos(n\omega_0 t)dt$$
$$b_n = \frac{2}{T}\int_0^T f(t)\sin(n\omega_0 t)dt$$

We will also show a new way to represent it.

Given some Fourier series of a function (neither even nor odd), there will be a corresponding cos and sin term with the same frequency. We can thus rewrite the Fourier series in *amplitude-phase form* as:

$$f(t) = a_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t + \phi_n)$$

... where:

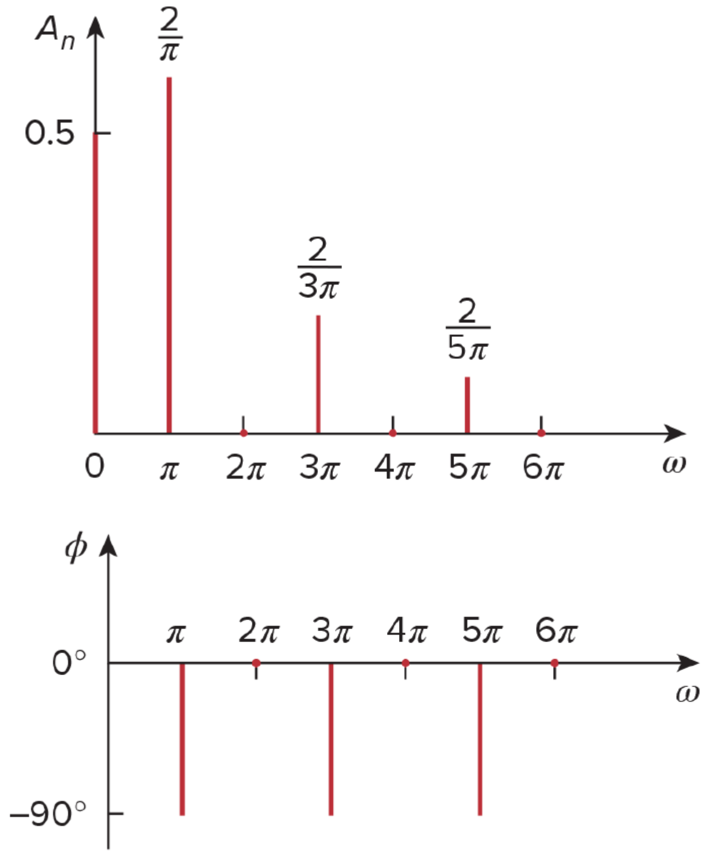
$$A_n = \sqrt{a_n^2 + b_n^2}$$
$$\phi_n = -\tan^{-1}\left(\frac{b_n}{a_n}\right)$$

... which could also be thought of as a complex number:

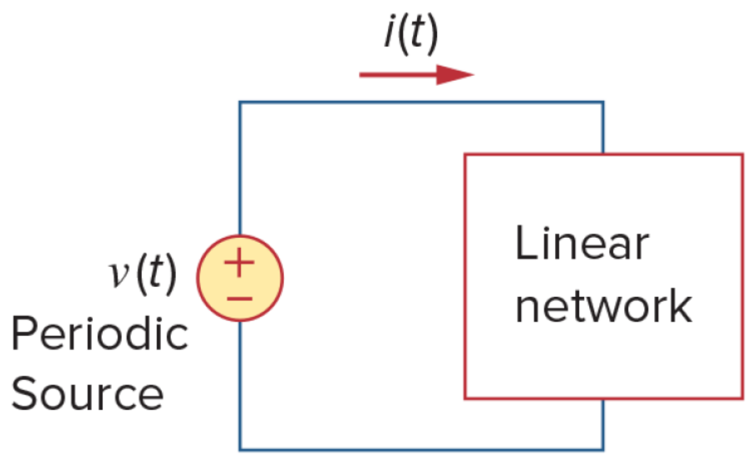
$$A_n = a_n - \jmath b_n = A_n \angle \phi_n$$

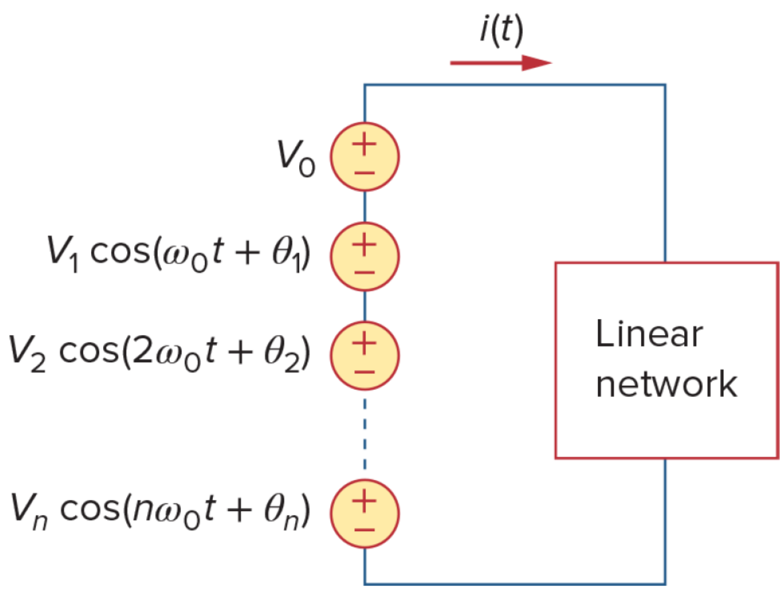
This allows us to combine the two terms into a single term.

The plots of the amplitudes and the phases of each term versus frequency is called the *frequency spectrum*, and the following is an example of one:



Now that we know this we can visually represent each term in the amplitude-phase form of the Fourier Series as an AC voltage source, as in:





17.5 Average Power and RMS Values

Given some periodic input in amplitude-phase form:

$$v(t) = V_{dc} + \sum_{n=1}^{\infty} V_n \cos(n\omega_0 t - \theta_n)$$

$$i(t) = I_{dc} + \sum_{m=1}^{\infty} I_m \cos(m\omega_0 t - \phi_m)$$

...the average power absorbed by the circuit (P) is:

$$P = V_{dc}I_{dc} + \frac{1}{2}\sum_{n=1}^{\infty} V_n I_n \cos(\theta_n - \phi_n)$$

By the definition of RMS, a function in amplitude-phase from has an RMS value of (F_{rms}):

$$F_{rms} = \sqrt{a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)}$$

The average power absorbed by a resistor with a periodic current input $f(t)$ is:

$$P = RF_{rms}^2$$

... or with a periodic voltage across it of $f(t)$:

$$P = \frac{F_{rms}^2}{R}$$

... or if we just choose a 1Ω resistor (known as *Parseval's Theorem*):

$$P_{1\Omega} = F_{rms}^2 = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

Chapter 19: Two-Port Networks

(Chapter not covered in F2021)

Conclusion

This concludes the content in this course. I hope these notes were helpful! Good luck in the exam!
- Adam Szava